

Of Atoms, Mountains, and Stars: A Study in Qualitative Physics

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One of the important achievements of physics in the present century is a fundamental understanding of the behavior and the properties of matter in the form in which it is found on the earth's surface. Almost all of the material phenomena which occur under terrestrial conditions are recognized as quantum mechanical consequences of the electric attraction between electrons and nuclei and of the gravitational attraction between massive objects. We should be able, therefore, to express all the relevant magnitudes which characterize the properties of matter in terms of the following six magnitudes: M , m , e , c , G , and h ; M is the mass of the proton, m and e are the mass and electrical charge of the electron, c is the light velocity, G is Newton's gravitational constant, and—most importantly— h is the quantum of action (1). In addition we will use the atomic number Z and the atomic weight A of the elements whose properties we study, since Ze determines the charge and AM the mass of the nucleus.

The books on atomic physics are full of detailed treatments of such questions. Quantum mechanics enables us to calculate the sizes, shapes, and energies of atoms and molecules; the varied properties of solids, liquids, and gases at different temperatures; and the emission, absorption, and scattering of light by matter. In this article we will discuss a few selected questions of this type. It will be done in a semiquantitative way, aiming only at the order of magnitude of the results, with the purpose of elucidating the essential concepts involved in such considerations

and demonstrating thereby the power and the fundamental simplicity of these insights. It will show how deeply the quantum is involved in our everyday experiences. It is important to realize, however, that such qualitative discussions never can replace actual quantitative calculations. They complement them and add to the understanding. Both approaches are necessary in order to understand nature. The quantitative treatment is needed to assure that all relevant aspects have been taken into account; the qualitative approach is needed to recognize the essential factors which govern the process and to get direct insights into the workings of the laws of nature.

The basis of our understanding is quantum mechanics. We start by formulating its fundamental principles in a simple way. Paul Ehrenfest, the famous physicist, once said that the laws of nature are "simple—but subtle." Quantum mechanics is based on the wave-particle duality in the nature of our fundamental entities, such as electrons, photons, or nuclei. It rests on the de Broglie relations, which connect the particle momentum p and the particle energy E with the wavelength λ and the frequency ω (2)

$$p = \frac{h}{\lambda}, E = h\omega \quad (1)$$

We will make use of the following three basic points of quantum mechanics:

1) If a wave motion is confined to a finite region in space, a series of patterns of standing waves emerge whose forms and frequencies depend on the

nature of the confinement. For example, an electron, confined by the Coulomb field of the nucleus, exhibits a series of wave patterns, which are shown in Fig. 1. Similar beautiful patterns develop in any confining field which is spherically symmetric. The simpler patterns have lower frequencies and, on the basis of the second de Broglie relation, lower energies. A new aspect came into physics when these patterns were found to be part of nature's structure. It is the appearance of characteristic shapes or forms in our interpretation of the fundamental structure of nature. Physics acquired a "morphic" character. Specific shapes had no justification in the classical physics of particles; quantum mechanics introduced this morphic trait, which is connected with the existence of discrete quantum states with well-defined properties. This is why quantum mechanics was needed to explain what nature tells us all over: our world is full of distinct and characteristic forms and shapes, from crystals to flowers.

2) Another property is connected with the confinement of a particle. The confined wave must have wavelengths comparable with or smaller than the linear dimensions of the confinement.

Since a maximum wavelength corresponds to a minimum momentum, a confined entity (for example, an electron) must have a kinetic energy, at least as big as the one that corresponds to the maximum wavelength. The first de Broglie relation leads to the following expression for the lowest kinetic energy $p^2/2m$

$$K_0 \sim \frac{h^2}{2mR^2} \quad (2)$$

where R is the linear dimension of the confinement and m the mass of the particle. Thus an electron cannot be completely at rest when it is confined. This effect can also be expressed in terms of a minimum pressure which the particle exerts on its confinement. It is a pressure tending to expand the confining region, since such an increase would lead to a decrease of

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kinetic energy. We will call it the "Schrödinger pressure," P_s , since it follows from the famous Schrödinger equation describing a particle-wave confined by a force. A particle, even at its lowest possible energy, will exert that pressure when confined. The pressure is the negative derivative of the energy with respect to the volume. Hence we get, omitting numerical factors,

$$P_s \sim \frac{h^2}{mR^5} \quad (3)$$

3) The third of our basic points is the Pauli principle. We express it in the following unconventional way: if more than one, say N , equal particles are confined in a volume V , the lowest kinetic energy of each particle is not that of a single electron (Eq. 2) but is higher. In fact, it is

$$K_0 \sim \frac{h^2}{2md^2}, \quad d = \left(\frac{V}{N}\right)^{1/3} \quad (4)$$

The length d is the linear dimension of a volume which is $1/N$ times the confining volume. That means that, if several equal particles are present, the characteristic confining volume for each particle is not the total volume but the volume divided by the number of particles. Each particle has its own "private" volume to which it is confined and which determines its minimum energy. Hence, when N equal particles are confined into a given volume V , the Schrödinger pressure is not N times larger than in the case of one particle, but $N^{5/3}$ times larger. We obtain

$$P_s \sim \frac{h^2}{m} \left(\frac{N}{V}\right)^{5/3} \quad (5)$$

When we are dealing with particles endowed with a spin, such as electrons or protons, "equal" particles must have their spins equally oriented. Only particles whose spins are all "up" or all "down" count as equal. Hence, a pair of particles with opposite spin are permitted to inhabit the same private volume. It will be useful to note the exact expression of the Schrödinger pressure exerted by n free electrons confined to volume V in their lowest energy state ($n/2$ with spin up, $n/2$ with spin down)

$$P_s = \frac{1}{2} (3\pi^2)^{2/3} \frac{h^2}{m} \left(\frac{n}{V}\right)^{5/3} \quad (5a)$$

The conventional way to formulate Pauli's great discovery uses a different formulation: not more than one particle of a given kind and spin is allowed to occupy a given quantum state. It is easy to show that our formulation is

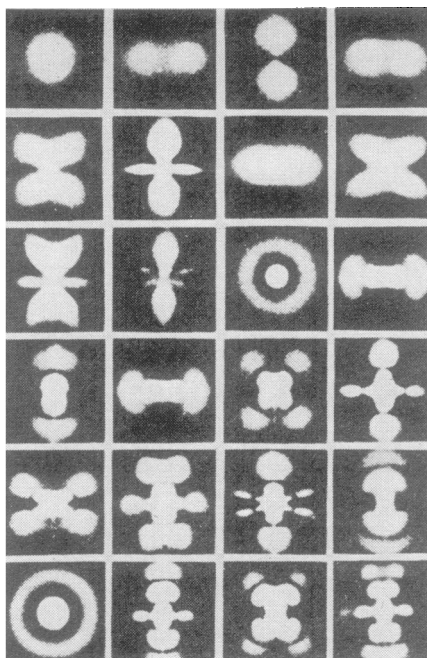


Fig. 1. Patterns of electron distributions in spherically symmetric fields. These are photographs of models. [From Weisskopf (10)]

equivalent. If the Pauli principle did not hold, all electrons would be allowed to be in the lowest quantum state. That would mean that the ground states of all atoms would be similar: the atomic electrons would all assemble in the lowest and simplest quantum state. All atoms would exhibit essentially the same properties, a most uninteresting world. We owe the variety of nature largely to the exclusion principle.

Atomic, Molecular, and Nuclear Relations

We now apply our considerations to the simplest atom, the hydrogen atom. Here an electron is confined by the electrostatic attraction of the proton. The electron wave may assume one of the different patterns of Fig. 1. Let us see what determines the size and energy of the lowest pattern. A balance is established between two countervailing effects—the electric attraction and the Schrödinger pressure toward expansion. In terms of energy this balance expresses itself as follows: the total energy of the lowest state is the sum of the potential energy ($-e^2/R$) of the attraction and the kinetic energy K_0

$$E = -\frac{e^2}{r} + \frac{h^2}{2mr^2} \quad (6)$$

The countervailing effects are balanced if this energy is a minimum. It is easily seen that the minimum is reached when the kinetic energy is equal to one-half of the absolute amount e^2/R of the potential energy. This is an example of the well-known virial theorem. This ratio between kinetic and potential energy is valid whenever the attractive force has a $1/R$ dependence and when relativistic effects can be neglected. We will use this theorem later when we deal with gravitational forces in stars.

Thus the electric attraction balances the Schrödinger pressure when Eq. 6 is a minimum, which occurs when

$$r = \frac{h^2}{me^2} \equiv a_0, \quad E = -\frac{me^4}{2h^2} \equiv Ry \quad (7)$$

Thus we have determined the two fundamental magnitudes of atomic physics, the Bohr radius, a_0 , which gives the size of the atom, and the Rydberg unit, Ry , which determines the energy by which the electron is bound to the proton. The hydrogen atom is the simplest of all atoms. Still its size and binding energy are characteristic for all atoms. These two values establish the order of magnitude of all atomic dimensions and binding energies.

When two atoms come close together to a distance comparable to their size, the total energy changes as function of the distance, because of the ensuing deformation of the electronic orbitals. A bond is formed if the energy decreases to a minimum value at the bonding distance; no bond is formed if the energy increases. The former happens in most cases—in particular, when the spin directions of the outer electrons are opposed so that a pair of them can form a common orbital. Then both of them are attracted by either nucleus, and a binding results.

The forming of a common orbital (homopolar bond) is not the only mechanism that leads to a lower total energy when atoms approach each other. Another simple mechanism (ionic bond) occurs when one of the atoms has a loosely bound electron and the other has a strong affinity for an additional electron. The atoms then exchange an electron when brought together and the ensuing opposite ions attract each other and form a bond. The ions, however, are kept from penetrating each other by the increased Schrödinger pressure which occurs when the two electron clouds try to occupy the same volume.

There are many other mechanisms

that lead to bondings between atoms. All cases can be described as the forming of a common quantum state of the combined electrons and nuclei under the influence of their mutual electric attractions and repulsions, a quantum state whose energy is lower than the sum of the energies of the separated atoms.

We therefore expect that most atoms join with others to form molecules; at lower temperatures we expect the molecules or atoms to aggregate to solids and liquids. In all these cases the distance between neighboring atoms will be of the order of the atomic sizes, a_0 , and the binding energies of the order of atomic energies; we expect, however, the binding energy to be somewhat smaller than a Rydberg, since the molecular bonds or the bonds in liquids and solids represent only an adjustment of the electron waves to the new situation of neighboring atoms.

We now express our knowledge of atomic and molecular physics in the form of four semiquantitative relations referring to the ionization energy I of an atom (the energy necessary to remove one electron), the radius R of an atom or a simple molecule, the dissociation energy D of a molecule, and the binding energy B of an atom or molecule in a solid or liquid. These relations contain certain "fudge factors" α , f , β , and γ whose values do not differ from unity by more than a factor of 10; they are given here within large limits only. Also, these formulas apply only to very simple atoms, molecules, or solids and certainly not to those complicated substances which we find in organic matter.

$$\begin{aligned} I &= \alpha Ry & 1 > \alpha > 1/4 \\ R &= f a_0 & 6 > f > 1 \\ D &= \beta Ry & 0.5 > \beta > 0.2 \\ B &= \gamma Ry & 0.3 > \gamma > 0.1 \text{ (solids)} \\ & & 0.1 > \gamma > 0.05 \text{ (liquids)} \end{aligned} \quad (8)$$

This is all we will need to know about atomic, molecular, and solid state physics.

In order to show the strength of these semiquantitative considerations we estimate the size and the energy, not of atoms but of nuclei. In nuclei, the constituents, protons and neutrons, are held together not by the Coulomb force but by the nuclear force. Figure 2 shows a rough sketch of the nuclear force potential as a function of the distance, neglecting the dependence on spin and symmetry. It also contains (dashed line) the electrostatic potential that fits the attractive part of the

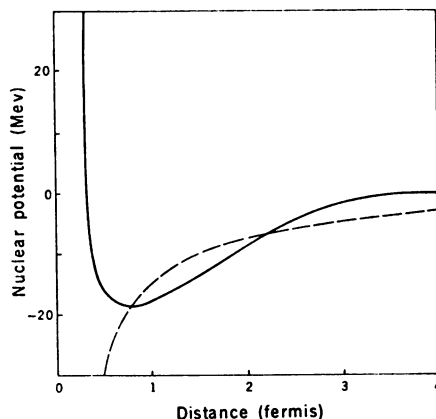


Fig. 2. Potential of the force between two nucleons. The solid curve is an approximate rendition of this potential. The exact value depends on the relative spin direction of the nucleons and on the symmetry of the quantum state. The dashed curve indicates the electrostatic potential between two particles of opposite charge, 3.3 times the charge of the electron. [From Weisskopf (10)]

nuclear potential best. It turns out that two nucleons attract each other, at least in that essential region where the attraction is important, as if they had opposite charges g of about 3 to 3.3 units. (The effective square of the charge, g^2 , is about ten times larger than e^2 .) The size and energy of a bound nuclear state should be of the order of the corresponding Bohr radius and Rydberg. We get the nuclear Bohr radius a_N and the nuclear Rydberg Ry_N by replacing e^2 by $g^2 \sim 10e^2$, and the electron mass m by the proton mass M

$$\begin{aligned} a_N &= \frac{h^2}{Mg^2} = a_0/20,000 \\ Ry_N &= \frac{Mg^4}{h^2} = 200,000 Ry \end{aligned}$$

Indeed, these are the dimensions and energies of nuclei. So much about nuclear physics.

Density and Hardness of Matter

Armed with this knowledge we are able to answer a number of questions of the type, Why just so big?, which may occur to anyone when contemplating nature. A hiking trip into the mountains is most conducive to posing simple questions of that kind. Some of those which will be answered in this article are (3)

- 1) Why is a piece of rock as heavy as it is?
- 2) Why is a rock or any other solid as hard as it is?

3) Why are the mountains as high as they are?

4) Why do we see solid or liquid objects clearly and distinctly through the atmosphere?

5) Why is the sun as big as it is?

We now turn to the first question. What is the density of compact matter like metal or rock? It is the weight per unit volume, or the weight of the atom or molecule divided by the volume which it fills. In compact matter atoms or molecules are contiguous; hence the density ρ is roughly given by

$$\rho = \frac{AM}{f^3 \frac{4\pi}{3} a_0^3} \quad (9)$$

where A is the atomic or molecular weight, M the mass of the proton, a_0 the Bohr radius, and f one of the factors defined in Eqs. 8. This factor depends somewhat on A ; the larger A , the larger the atomic or molecular radius. As a rough approximation $f = 1.5A^{1/5}$ serves as a good fit, although there are large fluctuations around this value, and these fluctuations enter into our formula in the third power. We then obtain the numerical result

$$\rho \sim 0.8A^{3/5} \text{ g/cm}^3$$

as an indication of the density of compact matter, roughly ranging between 1 and 10 g/cm³.

We now deal with the second question: the hardness of solids. Nothing is infinitely hard; every solid can be compressed if enough pressure is applied. The resistance against compression is expressed in terms of the "bulk modulus" C which is the ratio between the pressure P and the relative change of volume $\Delta V/V$; $C = PV/\Delta V$. It has the dimension of a pressure; indeed, a pressure of n percent of C is required to reduce the volume by n percent. Most of the ordinary solids have bulk moduli between 10^{11} and 10^{12} erg/cm³, which means that 10^5 to 10^6 atmospheres are needed to reduce appreciably the volume of a piece of metal or rock. There must be something fundamental about this fact, and we should be able to express this number in terms of our fundamental constants.

Let us look at a metal as an example of a solid. It consists of a lattice of positive ions with a lattice distance d of a few Bohr radii. In this lattice negative electrons are moving about almost freely as an "electron gas." In simple metals like sodium, copper, and silver there is one electron per ion in the gas, and the ions carry one posi-

tive charge. We now apply the concept of Schrödinger pressure to the electron gas, and remember that the pressure is determined by the number n of electrons in a given volume V according to Eq. 5a. Since we have one free electron per atom, the ratio n/V is equal to the reciprocal atomic volume, that is, the total volume of a piece of metal divided by the number of atoms. Clearly the cube root of the atomic volume $d = fa_0$ must be of the order of an atomic size. The values of f for copper, silver, and sodium are 4.29, 4.86, and 6.45. The Schrödinger pressure of the electron gas in these metals can be computed from Eq. 5a and one obtains $P_s = 3.8, 2.1,$ and 0.50 in units of 10^{11} erg/cm³, which are pressures of the order of 10^5 atmospheres. Actually these figures are underestimates (except in the case of sodium) since the deviations from a free electron gas have an effect similar to that of a smaller free volume.

This enormous pressure does not drive the metal apart because it is counteracted by the electric attraction between the ions and the electrons. The two effects are balanced in the metal under normal conditions. If one compresses the metal one acts against the Schrödinger pressure. We therefore expect the resistance to compression to be of the order of that pressure. Hence the bulk modulus must be of the same order as the Schrödinger pressures calculated above (4). Indeed the observed bulk moduli of the three metals are 14, 11, and 0.65 in units of 10^{11} erg/cm³. In the case of copper and silver those values are higher than the calculated ones. In these elements the ions fill a good part of the space and resist compression themselves; furthermore, the effective volume of the electron gas is considerably smaller than the atomic volume. These two circumstances add to the resistance against compression. In sodium, the free electron gas is a good approximation.

Let us now see how to determine the bulk modulus of a mineral like rock salt. It consists of a lattice of chlorine and sodium ions with a distance $d = fa_0$ between closest neighbors and $f = 5.32$. The oppositely charged ions are kept apart by a Schrödinger pressure which arises when the electron clouds of the ions begin to overlap. Even a relatively small overlap produces a rather large pressure because of the high electron density in the clouds. Under normal conditions this pressure P_s is held in equilibrium by the electric attraction

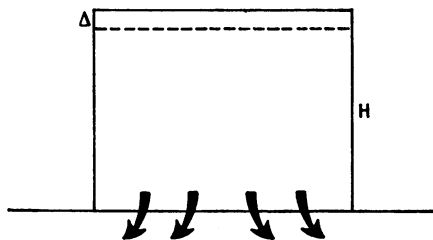


Fig. 3. The sinking of the mountain by the amount Δ corresponds to the displacement of a layer of thickness Δ from the top into the ground, and to a plastic flow of a comparable volume in the ground.

of the ions, and that gives us a means of calculating it. The electric energy per ion is known to be $-\mu e^2/v^{1/2}$, where $v = d^3 = f^3 a_0^3$ is the atomic volume and $\mu = 0.88$ is the so-called Madelung constant. This energy gives rise to a counterpressure P_e which is the negative derivative with respect to the volume: $-P_e = \frac{1}{3} \mu e^2 / f^4 a_0^4$. It balances the Schrödinger pressure, and therefore we get $P_s = -P_e$. With the above value of f we obtain $P_s = 1.07 \times 10^{11}$ erg/cm³. Again we argue that in compression we have to counteract the Schrödinger pressure, which in this case increases strongly with decreasing lattice distance d , since the overlap between the electron cloud is a steep function of d . Hence the bulk modulus will be of the order of but somewhat larger than P_s (4). Indeed the actual value in rock salt is 2.4×10^{11} erg/cm³.

We now can express the bulk modulus of any simple solid in terms of our fundamental constants. It is of the order of

$$C \sim F \frac{Ry}{a_0^3} = F \frac{e^{10} m^4}{h^8}$$

where F is a numerical constant, which contains the factor f with a rather high negative power. Roughly speaking, the value of the bulk modulus corresponds to an atomic energy per atomic volume, that is, a few electron volts per a few angstroms cubed.

Height of Mountains, Length of Water Waves

We now turn to the third question: Why are the mountains as high as they are? We simplify a mountain as a block of silicon oxide resting on a plane surface composed of the same material (see Fig. 3). The mountain will be too high to be supported by the base when the weight of the block is so large that the base matter starts to flow; the

mountain will sink when plastic deformation sets in.

Let us call H the height of our block at which it will begin to sink. Then H will be the maximum possible height which a mountain can reach. On a planet where the tectonic and vulcanic activities are high, as on the earth, the mountains will be of the same order of magnitude as H but somewhat smaller (5).

It should be pointed out that this problem is different from the previous one, which dealt with the compressibility of matter. The weight of the mountain indeed compresses the soil, but this compression does not reduce the height of the mountain in any appreciable way. There is no compression involved with plastic flow; it occurs without any important change in density. The energies involved in it are less than in compression.

At what height H will the mountain begin to sink into the bottom? It will happen at that height at which the energy gained by letting the mountain sink is equal to the energy necessary to engender plastic flow. This energy is comparable to and probably somewhat lower than the energy necessary to melt the rock. For our purposes, we will set these energies equal. The amount of matter to be "liquefied" is roughly the same as the amount of mountain matter which sinks into the ground. Hence, roughly speaking, the amount of gravitational energy gained by lowering matter from a height H must be equal to the liquefaction energy of the same amount of matter. The calculation can be done for each molecule separately. The mass of the molecule is A proton masses, where A is about 50, the atomic number of silicon oxide.

We then can write

$$AMHg = E_1$$

Here g is the gravitational acceleration on the earth and E_1 is the liquefaction energy per molecule. The latter energy must be a small fraction of the binding energy B of the material: $E_1 = \xi B$. After all, in the process of liquefaction the binding is not broken, only its directional stiffness is removed. How big is ξ ? We recall that melting of 1 g of ice costs 80 calories; the energy necessary to vaporize ice amounts to an additional 540 calories. Thus the factor ξ for ice is roughly 0.1. In metals and minerals this factor is of the order of 0.05. We therefore put $E_1 = \xi B = \xi \gamma Ry$ and with $\xi = 0.05$ and $\gamma = 0.2$

we get for the maximum height of a mountain

$$H = \xi\gamma \frac{Ry}{AMg} \sim 26 \text{ km} \quad (10)$$

The actual value is of the order of 10 km. This result is quite reasonable considering the fact that the energy necessary to produce plastic flow should be somewhat less than the liquefaction energy.

The formula for H is not yet in the form it should be in. The constant g is not really a fundamental number, and it can be expressed in terms of Newton's constant G and the number N_E of nucleons (protons and neutrons) in the earth, as follows:

$$g = \frac{GN_E M}{R_E^2}$$

where R_E is the earth's radius. Since the earth consists of "compact" matter (the molecules touch each other), R_E can be expressed in terms of the molecular or atomic radius R of SiO_2 or iron, the two substances which are typical for the composition of the earth. The molecular weight of SiO_2 fortunately is close to the atomic weight of iron ($A \approx 50$). Using Eqs. 8 we put

$$R_E \sim \frac{N_E^{1/3}}{A} R = \frac{N_E^{1/3}}{A} f a_0$$

We would like to express the final result not in terms of centimeters, but in terms of the only dignified unit of length in the atomic world: the Bohr radius. Putting everything together we obtain the maximum height of mountains

$$\frac{H}{a_0} \sim \xi\gamma f^2 \frac{\alpha}{\alpha_G} \frac{1}{N_E^{1/3}} \frac{1}{A^{5/3}} \quad (11)$$

Here α is the fine-structure constant and α_G is what one could call the gravitational fine-structure constant

$$\alpha = \frac{e^2}{hc} = 137^{-1}$$

$$\alpha_G = G \frac{M^2}{hc} = 5.88 \times 10^{-39}$$

where M is the mass of the proton. With the values $\xi = 0.05$, $\gamma = 0.2$, and $f = 3$, this expression gives the same result as before: $H/a_0 \approx 4.7 \times 10^{14}$. The ratio $\alpha/\alpha_G = 1.24 \times 10^{36}$ is the ratio of the electric and gravitational forces between two protons. This ratio enters in the above expression because the maximum mountain height results from a competition between the rigidity of rock and the force of gravity. The former is an effect of the electric forces between electrons and nuclei. Of course, the enormously large value of

α/α_G is compensated by the large numerical value of $N_E^{1/3}$.

Equation 11 leads to the approximate conclusion that the height of mountains on the other planets is reciprocally proportional to $N_P^{1/3}$ where N_P is the number of nucleons in the planet. If the rock constitution and geologic activity were the same as on the earth, the mountains would be about four times higher on the moon and about twice as high on Mars. The mountains on Mars indeed are about twice as high. The mountains on the moon, however, are only a little higher; they have not reached their maximum possible height because there is very little tectonic activity on the moon. One may also speculate as to the size of a planet on which the height of the mountains could be about the same as the radius. Such an object would be able to maintain in its rigid state a non-spherical shape, in spite of gravity. It can easily be seen from our relations that the maximum radius for this object is the geometric mean between the maximum mountain height on the earth and the earth's radius, that is, roughly 300 km. It is comforting to observe that the linear size of Deimos—a moon of Mars that is definitely not spherical—is smaller than this; it is of the order of 20 km.

Let us look for a moment at a seemingly very different natural phenomenon: water waves on the surface of a lake. When a light breeze starts to blow over a quiet water surface, the wavelength λ of the initial waves is of the order of a centimeter (λ is the actual wavelength divided by 2π). It would lead too far to enter into the physics of wave production; suffice it to say that the wind transfers its energy first to those waves whose propagation velocity is lowest. There is a certain wavelength λ_m for which the propagation velocity reaches a minimum, and this is roughly the wavelength that is first excited by wind. The minimum occurs because for larger waves the effect of the gravitational force increases, whereas for smaller ones the effect of the surface tension increases; both act as a restoring force in the wave oscillations. Hence the minimum wave velocity represents a balance between surface tension and gravity (6).

The expression for the velocity v of water waves is given by

$$v = \left(g\lambda + \frac{\sigma}{\lambda\rho} \right)^{1/2}$$

where λ is the wavelength, σ is the surface tension, and ρ is the density of water. The minimum velocity is reached when

$$\lambda = \left(\frac{\sigma}{g\rho} \right)^{1/2}$$

The surface tension σ is an energy per unit area. Let us express it in terms of the surface energy E_s contained in a surface area which involves only one surface molecule. This area is roughly π times the square of the radius $f a_0$ of the molecule; hence $\sigma \sim E_s/\pi f^2 a_0^2$. The surface energy E_s comes from the fact that the molecule at the surface is somewhat less strongly bound than a molecule in the bulk of the substance. The energy E_s is the difference between the binding energy B in the bulk and the binding energy B_s at the surface: $E_s = B - B_s$. Let us put $E_s = \xi' B$; in simple liquids E_s is about one-sixth of B . That relation is plausible if one considers the binding between molecules as six rectangular bonds originating at each molecule. At the surface one of them is inoperative. We then can write $\sigma = E_s/\pi f^2 a_0^2 = \xi' \lambda R y / \pi f^2 a_0^2$ and $\rho = 3AM/4\pi f^3 a_0^3$, and we get

$$\lambda^2 = \frac{4}{3} \gamma f \xi' \frac{R y a_0}{AMg}$$

A comparison of this expression with Eq. 10 shows the surprising result that $(\lambda/a_0)^2 \sim H/a_0$, apart from the difference between ξ' and ξ ($1/6$ and $1/20$) and the factor $\frac{1}{3} f$. Also, the values of γ and M are somewhat different in water and silicon oxide. The size of the wavelets on the lake is of the order of the square root of the height of mountains, if expressed in units of Bohr radii! This intriguing result comes from the fact that the surface tension, expressed in suitable units, is a fraction of the binding energy, of the same order of magnitude as the liquefaction energy. The length λ also is the result of a competition between atomic electric forces and gravity.

Visibility of Compact Matter

The fourth question—Why do we clearly see compact matter?—has to do with the interaction of light with matter. Light is an electromagnetic wave, and the electric part of the light field is the decisive one in affecting the atoms. In general the frequency of visible light is below the resonance frequencies of simple atoms and mole-

cules. These resonances lie mostly in the ultraviolet. Under these conditions, the effect of the oscillating electric light field on the atoms can be deduced from the effect of a static field. A static field \mathcal{E} induces a dipole moment D in an atom

$$D = p\mathcal{E} \quad (12)$$

The proportionality constant p —the polarizability—has the dimension of a volume.

We estimate the order of magnitude of p by the following simple consideration. Assume the atom is hydrogen, with a proton as a nucleus and an electron cloud of radius R . Let us estimate how strong a field is needed to displace the electron cloud relative to the nucleus so much that the nucleus is moved to the rim of the cloud. The dipole moment would then be $\bar{D} = eR$. The force which would drive the nucleus back to the center is e^2/R^2 ; hence we would need an electric field $\bar{\mathcal{E}} = e/R^2$ to hold the atom in this unusual state with the nucleus at the rim. If we assume that Eq. 12 is still right—this will be only approximately true—we get

$$p \sim \frac{\bar{D}}{\bar{\mathcal{E}}} = R^3 \quad (13)$$

We find that the polarizability p is of the order of the volume of the atom, a result which approximately holds also for more complicated atoms.

Let us now study what the effect of an incident light wave would be on an array of atoms arranged such that the average distance between the nearest neighbors is d . The value of d may be similar to or larger than $2R$. Each atom will acquire a dipole moment $D = p\mathcal{E}_0$ where \mathcal{E}_0 is the electric field strength of the light. The dipoles produce electric fields for themselves. What is the average strength \mathcal{E}_s of these secondary fields within the material? There is a well-known relation which says that the field \mathcal{E}_s produced by a uniform dipole density P is $\mathcal{E}_s = 4\pi P$. In our case the dipole density is given by $P = D/d^3$ so that we get

$$\mathcal{E}_s = 4\pi(p/d^3)\mathcal{E}_0$$

The secondary fields correspond to the secondary light emitted by the atoms when the incident wave excites them to perform vibrations. The secondary light waves interfere with the incident one, causing certain modifications. The incident light wave will be strongly modified by the secondary fields only if $\mathcal{E}_s \sim \mathcal{E}_0$, that is, if $4\pi p/d^3 \sim 1$

or if $d \sim 2R$. Thus only compact matter, where the distance between atoms is of the order of their diameter, strongly modifies incident light; the index of refraction (its real or imaginary part) differs from unity by an appreciable amount. These modifications manifest themselves in refraction or absorption within the first few wavelengths when light enters matter. They make compact matter plainly visible, in contrast to dilute matter such as normal air or colored vapors, where refraction is minimal and absorption takes place only over distances of many wavelengths.

The plain visibility of compact matter can be traced to the fact that the light field and the force that keeps the atoms together are both of the same nature, namely, electric. Let us see what would happen if the binding in the atom were caused by a different and stronger force. Let us assume, for example, that the charge responsible for the binding is $e' > e$, whereas the electric charge is still e . Then the polarizability of an atom would be smaller—the atom is stiffer—and we would get $p \sim (e/e')^2 R^3$. The secondary field \mathcal{E}_s would be $\mathcal{E}_s \sim 4\pi(e/e')^2 (R/d)^3 \mathcal{E}_0$, and even in compact matter \mathcal{E}_s would be smaller than \mathcal{E}_0 . Compact matter would be rather transparent to light. There is an example of such a situation in nature: nuclear matter. As we saw earlier, the nuclear attraction is about ten times stronger than the electric force between two charges. Indeed, light is scattered rather weakly when it passes through nuclei; it would have to penetrate many wavelengths into nuclear matter before being scattered or absorbed appreciably (7).

Size of a Star

We now come to our last question: Why is the sun as big as it is? One of the most amazing results of modern astronomy is the fact that the masses of stars with not too abnormal properties must lie between rather narrow limits, within a factor of a few hundred. The number N^* of nucleons in a star is given by a very simple combination of our fundamental constants

$$N^* = sN_0, \quad N_0 = \left(\frac{hc}{GM^2}\right)^{3/2} = 2.21 \times 10^{57} \quad (14)$$

Here s is a numerical factor which lies between 0.1 and ~ 60 . Indeed, the solar mass M_0 is equal to $N_0 M$ within a factor of 2: $M_0 = 0.54N_0 M$.

We are now going to derive this amazing relation. We assume with good justification that a star consists mostly of hydrogen gas, that is, of N^* protons and electrons. In our highly simplified arguments we will consider a star to be a sphere of radius R with uniform density and temperature. The temperature T is assumed to be high enough that the hydrogen atoms are ionized; electrons and protons are considered to be moving as free particles. It is gravity which keeps them together in the sphere. In order to connect the temperature with the effect of gravity we will make use of the previously mentioned virial theorem. It tells us that the time average of the kinetic energy (E_K) must be equal to one-half of the absolute amount of the time average of the potential energy (E_P). The gravitational potential energy is always a negative quantity; when masses get together, the potential energy drops. The virial theorem contains the absolute amount of the potential energy, which is a positive quantity

$$E_K = \frac{1}{2}|E_P| \text{ (virial theorem)} \quad (15)$$

The kinetic energy of the electrons and protons is mostly due to thermal motion. The equipartition theorem tells us that it is $(3/2)kT$ per particle, where k is the Boltzmann constant. The total kinetic energy should then be $3N^*kT$. The gravitational potential energy of two masses M at a distance r is $-GM^2/r$. In the average the distance between protons in the star is of the order of R and there are N^{*2} pairs of protons. (The potential energy of the electrons is negligible because of their small mass.) Hence the order of magnitude of the potential energy is $-GM^2N^{*2}/R$. The virial theorem then becomes

$$3N^*kT \sim \frac{1}{2} \frac{GM^2N^{*2}}{R} \quad (16)$$

We can draw several interesting conclusions from Eq. 16. The total energy E (the sum of the kinetic and potential energies) of the star, according to Eq. 15, is equal to the negative of the kinetic energy

$$E = -E_K = -3N^*kT \sim -\frac{1}{2} \frac{GM^2N^{*2}}{R} \quad (17)$$

The star, being hot, loses energy by radiation into the void; its energy must steadily become more negative. A lower energy means a higher temperature and also a smaller radius or stronger compression, according to Eq. 17. The loss of energy makes the star

hotter! The specific heat of a star is negative.

This paradoxical result is connected with another paradox of gravity. Imagine that a space traveler in a rocket going around the sun pulls a "brake" (loss of energy). One way of doing it would be to open an immense parachute which exerts a small frictional force in the dilute gas of meteorites or other interstellar matter. As a consequence of this braking action the rocket will move faster and not slower! The reason, of course, is that any change in kinetic energy is necessarily connected with a double change of potential energy in the opposite direction (virial theorem). The braking accelerates the rocket by making it fall toward the sun, as it were. This is in complete analogy to the fact that energy loss by radiation heats up the star; the concomitant compression corresponds to the falling toward the center.

Let us now transform the virial relation (Eq. 16) to a form which is more convenient. It is more practical to replace the radius R with another length d , which is equivalent to the average distance between neighboring protons in the star. It is the linear dimension of the volume which each proton (or electron) occupies for itself

$$d = R/(N^*)^{1/3} \quad (18)$$

Then, after we have divided by N^* , Eq. 13 becomes

$$3kT \sim \frac{1}{2} \frac{GM^2 (N^*)^{2/3}}{d} = \frac{1}{2} \frac{GM^2}{hc} (N^*)^{2/3} \frac{hc}{d}$$

Here the left side is the kinetic energy per proton and the right side is half the amount of the potential energy per proton. This can be written in the simple form

$$kT = f \left(\frac{N^*}{N_0} \right)^{2/3} \frac{hc}{d} \quad (19)$$

where N_0 is the number defined in Eq. 14 and f is a numerical constant of order unity which, in this rough approximation, will be dropped henceforth.

We return to the fact that a star constantly loses energy by radiation, and that this causes a rise in temperature T and a decrease in size d . There is a limit to this rise, however. If the temperature reaches a certain value which we may call the "nuclear ignition temperature," T_{ig} , the nuclear fire starts burning, hydrogen is transformed to helium. Here we meet a

similar paradox. The nuclear fire does not raise the temperature, it keeps the star cool. The reason is simple to see. The nuclear reactions replace the energy lost by radiation, and the total energy no longer diminishes but stays constant and so does the temperature (see Eq. 16). The star functions like a thermostat: if the nuclear energy production becomes too large the star expands, which cools it down and reduces the nuclear fire.

We are now able to determine a minimum value for the mass of a star. It is connected with another factor which limits the rise of the temperature by radiation loss. It is the Schrödinger pressure of the electrons in the star. When the distance d between electrons becomes small, the Schrödinger pressure rises and may become equal to the gravitational pull; then the compression ceases and so does the increase in temperature. In other words, the matter of the star resists compression essentially in the same way in which a chunk of metal or any other solid material resists it, that is, by the effect of the Schrödinger pressure of the electrons.

In the star the effect manifests itself in the following way: with decreasing distance d between electrons the minimum kinetic energy (Eq. 4) of the electrons increases; this energy must be provided for by the gravitational energy gained when the star contracts. Hence that gain can no longer be used to increase the temperature. Indeed, there comes a moment when the contraction ceases altogether since, according to the Pauli principle, the minimum kinetic energy of the electrons increases as $1/d^2$ (see Eq. 4), whereas the gravitational energy gained by contraction increases as only $1/d$ (8). We therefore obtain a condition for stopping the contraction and temperature increase even if no nuclear reactions take place. They will stop when the gravitational energy per electron is of the same order as the minimum kinetic energy (Eq. 4). The gravitational energy per electron is the same as that per proton, and its order of magnitude is given by the right side of Eq. 19. We therefore get a condition for the distance d_{stop} at which contraction stops

$$\left(\frac{N^*}{N_0} \right)^{2/3} \frac{hc}{d_{stop}} \sim \frac{h^2}{2m(d_{stop})^3}$$

or (the factor 2 is omitted)

$$\frac{1}{d_{stop}} \sim \left(\frac{N^*}{N_0} \right)^{2/3} \frac{mc}{h}$$

In principle, this determination of d_{stop} is the same as the determination of the Bohr radius a_0 of the hydrogen atom from Eq. 6. There, it was the balance between the electric attraction between proton and electron and the Schrödinger pressure which determined a_0 ; here, it is the balance between the gravitational attraction of all N^* protons and the Schrödinger pressure which determines d_{stop} .

The maximum temperature T_{max} which the star can reach before its contraction is stopped by the Schrödinger pressure of the electrons is given by Eq. 19, which connects d with T

$$kT_{max} \sim \left(\frac{N^*}{N_0} \right)^{2/3} \frac{hc}{d_{stop}} \sim \left(\frac{N^*}{N_0} \right)^{4/3} mc^2 \quad (20)$$

where m is the electron mass. What is the minimum number N^* of protons in a star? We understand by the term "star" an assembly of matter that produces nuclear reactions in the interior in order to replace its radiation losses and thus attain a certain degree of stability. The minimum value of N^* must be such that the maximum reachable temperature is higher than the ignition temperature T_{ig} of nuclear reactions. The ignition temperature is the one for which the kinetic energy ϵ of the protons is high enough that, in a collision between two protons, the two particles are able to merge in spite of the electrostatic repulsion. Quantum mechanics teaches us that energy ϵ is

$$\epsilon \sim \frac{\pi^2}{2} \frac{Me^4}{h^2} = 123,000 \text{ ev}$$

The temperature T_{ig} necessary to attain this corresponds to

$$kT_{ig} = \eta \epsilon$$

where η is a numerical factor smaller than unity since the Maxwell distribution always provides for protons of higher energy than kT (9). Clearly T_{max} must be larger than T_{ig} , and we get from Eq. 20

$$\frac{N^*}{N_0} > \left(\frac{\eta \epsilon}{mc^2} \right)^{3/4} = \left(\frac{\pi^2 M e^4}{2 m h^2 c^2 \eta} \right)^{3/4}$$

The number of protons in a star must be at least $0.58\eta^{3/4}$ times the number N_0 . A more accurate calculation gives $0.1N_0$ as the lower limit. Thus, our sun with $N^* = 0.54N_0$ is not much larger than the minimum. It is a relatively small starlet.

Is there an upper limit of N^* ? There is indeed, and it comes from the fact that a heavy star gets very hot

and the ensuing radiation pressure makes it unstable. In order to determine the upper limit of N^* we must amplify our previous considerations in two respects.

First, the kinetic energy within the star is not only the energy of the motion of protons and electrons; light quanta also have "kinetic" energy—it is $h\omega$ per quantum—and it appears as the energy of the thermal radiation in the star. We will soon see that this energy is not an important contributor if N^*/N_0 is smaller than or comparable to unity. Therefore, it was justified to forget it in our previous considerations.

Second, the virial theorem in the form of Eq. 15 is only correct for nonrelativistic particles. If some of the particles are extremely relativistic, as light quanta are, we must write instead of Eq. 15

$$E_k = \Lambda |E_p| \quad (29)$$

where Λ is between $\frac{1}{2}$ and 1. If most particles are extremely relativistic, the factor Λ becomes unity. Since the potential energy is negative, the condition $\Lambda = 1$ implies that kinetic and potential energy are equal and opposite; the total energy, therefore, would become zero. This would mean that the star would not hold together (a bound system must have negative total energy). This is another way of saying that the light pressure destabilizes the star if the radiation energy is much larger than the kinetic energy of the protons and electrons.

Let us now figure out when the radiation energy becomes larger than the kinetic energy of the particles. The latter is of the order of kT per particle; the former can be determined from the well-known Stefan-Boltzmann law which says that the radiation energy density u at a given temperature is proportional to the fourth power of the temperature

$$u = \sigma (kT)^4, \sigma = f \frac{1}{h^3 c^3} \quad (23)$$

The proportionality constant σ is simply $(hc)^{-3}$ apart from a constant f very near to unity. We will put $f = 1$.

A simple way of understanding this relation is to recognize that, in thermal equilibrium, the average energy of the light quanta must be $\sim kT$, and that there is about one quantum in a volume whose linear dimensions are of the

order of the wavelength λ of these light quanta. Hence the energy density would be $u = kT/\lambda^3$, and $hc/\lambda = kT$, which corresponds to Eq. 23.

The radiation energy E_r in the star is the energy density multiplied by the volume

$$E_r = f \frac{(kT)^4 4\pi R^3}{h^3 c^3 3}$$

In order to keep the star stable, the radiation energy must not be too big compared to the kinetic particle energy $3N^*kT$, or

$$\frac{E_r}{3N^*kT} \sim \left(\frac{kT}{hc}\right)^3 \frac{R^3}{N^*} = \left(\frac{kT}{hc/d}\right)^3 = \text{"not too big compared to unity"} \quad (24)$$

Here numerical constants have been dropped.

Let us now rewrite the virial theorem (Eq. 16) including the radiation energy and the factor Λ :

$$E_r + 3N^*kT \sim \Lambda \frac{GM^2(N^*)^2}{R} \quad (16a)$$

We divide this by N^* and introduce the relations Eqs. 14 and 18, as we did to get Eq. 19 from Eq. 16.

$$(kT)^4 \left(\frac{d}{hc}\right)^3 + kT \sim \left(\frac{N^*}{N_0}\right)^{3/2} \frac{hc}{d} \quad (19a)$$

Here we put numerical factors such as Λ , 3, and $4\pi/3$ equal to unity. Let us call x the combination

$$x = \frac{T}{hc/d}$$

and we find that Eq. 19a is equivalent to

$$x^4 + x \sim \left(\frac{N^*}{N_0}\right)^{3/2} \quad (25)$$

The stability condition, Eq. 24, tells us that x^3 should not be too big compared to unity. From this it follows that $x^4 + x$ should not exceed unity by a large factor, and according to Eq. 25 neither should $(N^*/N_0)^{3/2}$. We therefore conclude that a star is stable as long as N^* is not very much larger than N_0 . Indeed, a more accurate calculation shows that N^* must be smaller than $60N_0$.

Thus it is seen that the number N^* of protons in a star must be smaller than a certain multiple of N_0 in order for them not to be blown apart by radiation, and larger than a certain fraction of N_0 in order for them to reach the temperature which ignites the nuclear fire. The quantity N_0 is a

simple combination (Eq. 14) of our fundamental constants. We thus have shown what we intended to do: the number N^* of protons in a star is roughly of the order of N_0 :

$$60N_0 > N^* > 0.1N_0$$

These are rather narrow limits considering that N_0 is as large as 2×10^{57} .

References and Notes

1. We always use $h = h^*/2\pi$, where h^* is the conventional value of Planck's constant; $h = 1.05 \times 10^{-27}$ erg/sec.
2. We use the symbol $\lambda = \lambda^*/2\pi$; ω stands for the frequency per $1/2\pi$ seconds: $\omega = 2\pi \nu$, where λ^* is the ordinary wavelength and ν is the ordinary frequency.
3. I was greatly helped in these considerations by discussions with S. Chandrasekhar, D. Hawkins, E. Purcell, E. S. Salpeter, V. Telegdi, P. Wolff, and by reading C. Kittel's book *Introduction to Solid State Physics* (Wiley, New York, 1966) and E. Salpeter's article "Dimensionless ratios and stellar structure" published in the volume honoring H. A. Bethe, *Perspectives in Modern Physics* (Wiley, New York, 1966). None of those persons should be blamed, however, for the risky approximations and generalizations made in this article.
4. This is seen in the following way: under normal conditions the Schrödinger pressure and the counterpressure (caused by the mechanisms of attraction) balance each other. When the volume is reduced by n percent the Schrödinger pressure increases by n_s percent and the counterpressure by n_c percent. Equilibrium exists only if $n_s > n_c$. It follows from the definition of the bulk modulus C that $nC = (n_s - n_c)P_0$. In general the ratio $(n_s - n_c)/n$ is of the order of unity or larger, but not by an inordinate factor.
5. There are other limitations to the height of mountains, such as the fact that mountain ranges usually are made of lighter material than the earth's crust. The mountains float, as it were. There is a limit to the height when the submerged part reaches down into the liquid layers of the crust. That limit, however, is of the same order of magnitude as the one treated in the text.
6. The same length λ appears also in a different context: the radius of a water drop hanging from a wet ceiling and being just big enough to detach and fall, is $3^{1/2}$ times the wavelength λ of a water wave with minimum velocity.
7. We are referring here not necessarily only to visible light, but to the appropriate "nuclear" light, namely, γ rays, which is the light "seen" (emitted and absorbed) by nuclei.
8. If the temperatures become very high, as is the case for massive stars, the kinetic energy of the electrons becomes much larger than their mass energy. Then one must use the relativistic expression for the kinetic energy, which for very high energies increases as $1/d$ only. Thus in these cases the Pauli principle is unable to stop contraction and we get collapse of the star into a "black hole." The discussion of these heavy stars is beyond the topic of this article.
9. Here we have greatly simplified the situation. The nuclear reactions can take place even if kT is considerably lower than the energy necessary to overcome the repulsion barrier, first, because the Maxwell distribution provides for protons with energies much higher than kT , and second, because there is a quantum mechanical penetration through the barrier. Indeed, the actual temperature in the center of the sun is such that kT is several kiloelectron volts, which means η is as small as 0.02. Our simplified approach serves to illustrate the logical reasoning. Together, with other simplifications it yields an approximate result for N^* that is not too far from that of the more exact calculations.
10. V. F. Weisskopf, *Science* **168**, 923 (1970).