Outline of lectures : refresher in many-body theory

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0.1 Prologue

Below, I make reference to the following free lecture notes. If you feel you are missing some prerequisites, everything is in these lecture notes.

http://www.physique.usherbrooke.ca/tremblay/cours/phy-892/N-corps.pdf

Many of these lectures are on YouTube

https://www.youtube.com/playlist?list=PL9IKDS79pLpNJS9KLrAZU0zw_Tin4E6ed

0.2 Lecture 1 (30 minutes) Second quantization

Chapter 81 : Handeling many-interacting particles : Second quantization

81.1 Fock space : Creation-annihilation operators

Number operator

81.2 Change of basis

87.2.1 Position and momentum basis

87.2.2 Wave functions

81.3 One-body operators

81.4 Two-body operators

0.3 Lecture 2 (45 minutes) Time-ordered product, Green functions

Chapter 83 Perturbation theory (interaction representation)

$$e^{-\beta\widehat{K}} = e^{-\beta\widehat{K}_0}\widehat{U}(\beta) \quad (1)$$

$$\widehat{U}(\beta) \equiv T_\tau\left[e^{-\int_0^\beta \widehat{K}_1(\tau)d\tau}\right] \quad (2)$$

$$\widehat{K}_1(\tau) \equiv e^{\widehat{K}_0\tau}\widehat{K}_1 e^{-\widehat{K}_0\tau}. \quad (3)$$

Chapter 29 Matsubara Green's function

84.1 Photoemission and fermion correlation functions

$$\frac{\partial^2\sigma}{\partial\Omega\partial\omega} \propto \sum_{mn} e^{-\beta K_m}\langle m|\, c^\dagger_{\mathbf{k}_{||}}\,|n\rangle\,\langle n|\, c_{\mathbf{k}_{||}}\,|m\rangle\,\delta\left(\omega-(K_m-K_n)\right) \quad (4)$$

29.1 Definition of the Matsubara Green function

$$\mathcal{G}_{\alpha\beta}(\tau) = -\left\langle T_\tau c_\alpha(\tau)\, c^\dagger_\beta(0)\right\rangle \quad (5)$$

$$= -\left\langle c_\alpha(\tau)\, c^\dagger_\beta(0)\right\rangle\theta(\tau) + \left\langle c^\dagger_\beta(0)\, c_\alpha(\tau)\right\rangle\theta(-\tau). \quad (6)$$

29.3 Antiperiodicity and Fournier expansion

$$\mathcal{G}_{\alpha\beta}(ik_n) = \int_0^\beta d\tau e^{ik_n\tau}\mathcal{G}_{\alpha\beta}(\tau) \tag{7}$$

$$\mathcal{G}_{\alpha\beta}(\tau) = T\sum_n e^{-ik_n\tau}\mathcal{G}_{\alpha\beta}(ik_n) \tag{8}$$

29.8 Green function for $U = 0$

$$\mathcal{G}_{\mathbf{k}}(ik_n) = \frac{1}{ik_n - \zeta_{\mathbf{k}}} \tag{9}$$

29.2 Time-ordering operator in practice

$$\left\langle T_\tau\psi(\tau_1)\psi^\dagger(\tau_3)\psi(\tau_2)\psi^\dagger(\tau_4)\right\rangle = -\left\langle T_\tau\psi^\dagger(\tau_3)\psi(\tau_1)\psi(\tau_2)\psi^\dagger(\tau_4)\right\rangle \tag{10}$$

0.4 Lecture 3 (45 minutes) Spectral weight, Self-energy, Quasiparticles

84.4 Spectral weight and how it is related to $\mathcal{G}_{\mathbf{k}}(ik_n)$ and to photoemission

$$\frac{\partial^2\sigma}{\partial\Omega\partial\omega} \propto A_{\mathbf{k}}(\omega) f(\omega) \tag{11}$$

29.5 Lehman representation

$$\mathcal{G}_{\mathbf{k}}(ik_n) = \int \frac{d\omega}{2\pi}\frac{A_{\mathbf{k}}(\omega)}{ik_n - \omega} \tag{12}$$

$$A_{\mathbf{k}}(\omega) \equiv \sum_{n,m}\frac{1}{Z}\left(e^{-\beta K_m} + e^{-\beta K_n}\right)\langle n| c_{\mathbf{k}} |m\rangle \langle m| c_{\mathbf{k}}^\dagger |n\rangle 2\pi\delta(\omega - (K_m - K_n)) \tag{13}$$

29.6 Obtaining the spectral weight from $\mathcal{G}_{\mathbf{k}}(ik_n)$, the problem of analytic continuation

$$\mathcal{G}_{\mathbf{k}}(ik_n) = \int \frac{d\omega'}{2\pi}\frac{A_{\mathbf{k}}(\omega')}{ik_n - \omega'} \tag{14}$$

$$G_{\mathbf{k}}^R(\omega) = \int \frac{d\omega'}{2\pi}\frac{A_{\mathbf{k}}(\omega')}{\omega + i\eta - \omega'} \tag{15}$$

The notion of self-energy, what it means, what it hides
(20 minutes) Chapter 17 Self-energy

$$A(\mathbf{k};\omega') = \frac{2\Gamma}{(\omega - \widetilde{\varepsilon}_{\mathbf{k}})^2 + \Gamma^2} \tag{16}$$

$$G^R(\mathbf{k},\omega) = \frac{1}{\omega - \widetilde{\varepsilon}_{\mathbf{k}} + i\Gamma}. \tag{17}$$

$$G^R(\mathbf{k},\omega) = \frac{1}{\omega + i\eta - \varepsilon_{\mathbf{k}} - \Sigma^R(\mathbf{k},\omega)} = \frac{1}{G_0^R(\mathbf{k},\omega)^{-1} - \Sigma^R(\mathbf{k},\omega)}. \tag{18}$$

With the simple approximation that we did for the self-energy,

$$\Sigma^R(\mathbf{k},\omega) = \widetilde{\varepsilon}_{\mathbf{k}} - \varepsilon_{\mathbf{k}} - i\Gamma, \tag{19}$$

$$G^R(\mathbf{k},\omega)^{-1} = G_0^R(\mathbf{k},\omega)^{-1} - \Sigma^R(\mathbf{k},\omega) \tag{20}$$

$$G^R(\mathbf{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{\omega - \widetilde{\varepsilon}_{\mathbf{k}} + i\Gamma} = -i\theta(t) e^{-i\widetilde{\varepsilon}_{\mathbf{k}} t - \Gamma t} \tag{21}$$

$$|\langle \mathbf{k}| \psi(t)\rangle|^2 = \left|G^R(\mathbf{k},t)\right|^2 = \theta(t) e^{-2\Gamma t}. \tag{22}$$

18.3 Importance of poles of $G^R_{\mathbf{k}}$, Dyson's equation

$$\mathcal{G}_{\mathbf{k}}(ik_n) = \mathcal{G}^0_{\mathbf{k}}(ik_n) + \mathcal{G}^0_{\mathbf{k}}(ik_n)\Sigma_{\mathbf{k}}(ik_n)\mathcal{G}_{\mathbf{k}}(ik_n) \tag{23}$$

$$G^R_{\mathbf{k}\uparrow}(\omega)^{-1} = G^{(0)R}_{\mathbf{k}\uparrow}(\omega)^{-1} - \Sigma^R_{\mathbf{k}\uparrow}(\omega) \tag{24}$$

85.3 A few properties of the self-energy

$$\operatorname{Im}\Sigma^R_{\mathbf{k}\uparrow}(\omega) < 0 \tag{25}$$

31.3 Some experimental results
31.4 Quasiparticules

$$A(\mathbf{k},\omega) \approx 2\pi Z_{\mathbf{k}} \frac{1}{\pi} \frac{-Z_{\mathbf{k}} \operatorname{Im}\sum^R(\mathbf{k},\omega)}{(\omega - E_{\mathbf{k}} + \mu)^2 + \left(Z_{\mathbf{k}} \operatorname{Im}\sum^R(\mathbf{k},\omega)\right)^2} + inc \tag{26}$$

31.5 Fermi liquid interpretation

$$\operatorname{Im}\Sigma^R_{\mathbf{k}\uparrow}(\omega) \sim \omega^2 + (\pi T)^2 \tag{27}$$

0.5 Lecture 4 (90 minutes) Coherent states for fermions

Chapter 79 Coherent states for fermions

79.1 Grassmann variables for fermions

$$c\left|\eta\right\rangle = \eta\left|\eta\right\rangle \;;\; \left|\eta\right\rangle = e^{-\eta c^{\dagger}}\left|0\right\rangle \tag{28}$$

79.2 Grassmann integrals

$$\int d\eta = 0\;;\; \int d\eta\eta = 1 \tag{29}$$

79.3 Change of variables in Grassmann integrals

$$\psi_i = \sum_{j=1}^{N} U_{ij}\eta_j \;;\; \prod_{i=1}^{N}\int d\psi_i\ F(\psi_i) = \det\left[U\right]^{-1}\prod_{k=1}^{N}\int d\eta_k\ F(\eta_k) \tag{30}$$

79.4 Grassmann Gaussian Integrals

$$\int \mathcal{D}\eta^{\dagger}\int \mathcal{D}\eta e^{-\boldsymbol{\eta}^{\dagger}\mathbf{A}\boldsymbol{\eta}-\boldsymbol{\eta}^{\dagger}\mathbf{J}-\mathbf{J}^{\dagger}\boldsymbol{\eta}} = \det\left(A\right)\exp\left(\mathbf{J}^{\dagger}\mathbf{A}^{-1}\mathbf{J}\right) \tag{31}$$

79.5 Closure, overcompleteness, trace formula

$$\mathrm{Tr}[O] = \int d\eta^{\dagger}\int d\eta e^{-\eta^{\dagger}\eta}\left\langle -\eta\right| O\left|\eta\right\rangle \tag{32}$$

Chapter 80 The coherent-state functional integral for fermions

80.1 and 80.2 A simple example with a single fermion

$$Z = \int \mathcal{D}\eta^{\dagger}\int \mathcal{D}\eta\exp\left(-S\right) \tag{33}$$

$$S = \int_0^{\beta} d\tau\left(\eta^{\dagger}\left(\tau\right)\frac{\partial}{\partial\tau}\eta\left(\tau\right) + \varepsilon\left(\tau\right)\eta^{\dagger}\left(\tau\right)\eta\left(\tau\right)\right) \tag{34}$$

80.3 Wick's theorem

$$\left(-1\right)^n\left\langle T_{\tau}c\left(\tau_n\right)c^{\dagger}\left(\tau_n'\right)\cdots c\left(\tau_2\right)c^{\dagger}\left(\tau_2'\right)c\left(\tau_1\right)c^{\dagger}\left(\tau_1'\right)\right\rangle \tag{35}$$

$$= \left(-1\right)^n\frac{1}{Z}\int \mathcal{D}\eta^{\dagger}\int \mathcal{D}\eta e^{-\boldsymbol{\eta}^{\dagger}\left(-\mathcal{G}^{-1}\right)\boldsymbol{\eta}}\eta\left(\tau_n\right)\eta^{\dagger}\left(\tau_n'\right)\cdots\eta\left(\tau_2\right)\eta^{\dagger}\left(\tau_2'\right)\eta\left(\tau_1\right)\eta^{\dagger}\left(\tau_1'\right)$$

$$= \det\begin{bmatrix} \mathcal{G}\left(\tau_1,\tau_1'\right) & \mathcal{G}\left(\tau_1,\tau_2'\right) & \cdots & \mathcal{G}\left(\tau_1,\tau_n'\right) \\ \mathcal{G}\left(\tau_2,\tau_1'\right) & \mathcal{G}\left(\tau_2,\tau_2'\right) & \cdots & \mathcal{G}\left(\tau_2,\tau_n'\right) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{G}\left(\tau_n,\tau_1'\right) & \mathcal{G}\left(\tau_n,\tau_2'\right) & \cdots & \mathcal{G}\left(\tau_n,\tau_n'\right) \end{bmatrix}. \tag{36}$$

0.6 Lecture 5 (90 minutes) Many-body perturbation theory

Chapter 87 Source fields for Many-Body Green's function

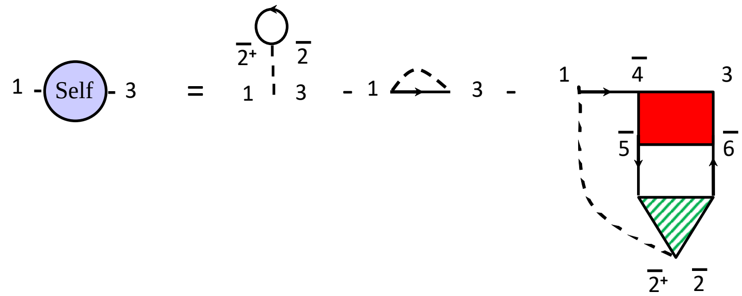


Figure 0-1 Diagrams for the self-energy. The dashed line represent the interaction. The first two terms are, respectively, the Hatree and the Fock contributions. The textured square appearing in the previous figure for the four-point function has been squeezed to a triangle to illustrate the fact that two of the indices (coordinates) are identical.

87.1 A simple example in classical statistical mechanics

$$\frac{\delta^2 \ln Z[h]}{\beta^2 \delta h(\mathbf{x}_1)\,\delta h(\mathbf{x}_2)} = \langle M(\mathbf{x}_1)\,M(\mathbf{x}_2)\rangle_h - \langle M(\mathbf{x}_1)\rangle_h \langle M(\mathbf{x}_2)\rangle_h \tag{37}$$

80.6 c-number source fields to generate fermion bi-linears

$$Z[\phi] = \int \mathcal{D}\psi^\dagger \int \mathcal{D}\psi \exp\left(-S - \psi^\dagger(\overline{1})\,\phi(\overline{1},\overline{2})\,\psi(\overline{2})\right) \tag{38}$$

$$-\frac{\delta \ln Z[\phi]}{\delta\phi(2,1)} \equiv -\left\langle T_\tau \psi(1)\,\psi^\dagger(2)\right\rangle_\phi = \mathcal{G}(1,2)_\phi\,. \tag{39}$$

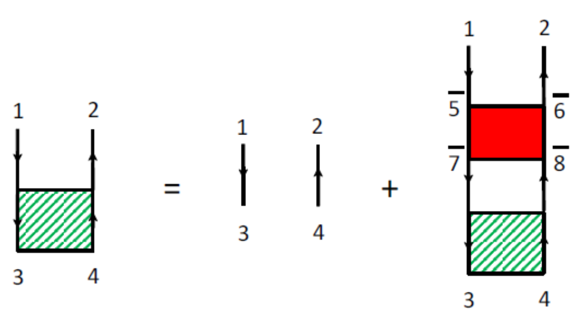
$$\frac{\delta\mathcal{G}(1,2)_\phi}{\delta\phi(3,4)} = \left\langle \psi(1)\,\psi^\dagger(2)\,\psi^\dagger(3)\,\psi(4)\right\rangle_\phi + \mathcal{G}(1,2)_\phi\,\mathcal{G}(4,3)_\phi \tag{40}$$

80.7 Schwinger-Dyson equation of motion from functional integrals

$$\left[\mathcal{G}_0^{-1}(1,\overline{2}) - \phi(1,\overline{2})\right]\mathcal{G}(\overline{2},2)_\phi = \delta(1-2) - V(1,\overline{2})\left\langle \psi^\dagger(\overline{2})\,\psi(\overline{2})\,\psi(1)\,\psi^\dagger(2)\right\rangle_\phi \tag{41}$$

$$\Sigma(1,\overline{2})_\phi\,\mathcal{G}(\overline{2},2)_\phi = -V(1-\overline{2})\left\langle \psi^\dagger\left(\overline{2^+}\right)\psi(\overline{2})\,\psi(1)\,\psi^\dagger(2)\right\rangle_\phi,$$

Chapter 36 Equations of motion for $\mathcal{G}$ in the presence of source fields
36.3 Four-point function fromfunctional derivatives



36.4 Self-energy from functional derivatives
Chapter 72 Luttinger Ward Functional
72.3 Luttinger Ward functional and the Legendre transform of $-T \ln Z[\phi]$

$$\Omega[\mathcal{G}] = F[\phi] - \mathrm{Tr}[\phi\mathcal{G}] \;;\; \frac{1}{T}\frac{\delta\Omega[\mathcal{G}]}{\delta\mathcal{G}(1,2)} = 0 \text{ in equilibrium} \tag{42}$$

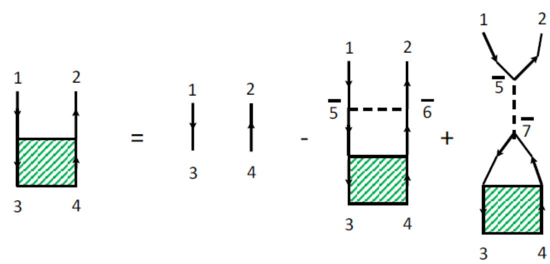
Chapter 76 The constraining-field method (?)
76.1 Another derivation of the Baym-Kadanoff functional

$$\Omega[\mathcal{G}] = \Phi[\mathcal{G}] - \mathrm{Tr}\left[\left(\mathcal{G}_0^{-1} - \mathcal{G}^{-1}\right)\mathcal{G}\right] + \mathrm{Tr}\left[\ln\left(\frac{-\mathcal{G}}{-\mathcal{G}_\infty}\right)\right] \tag{43}$$

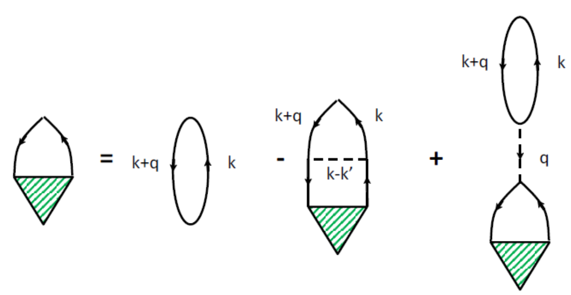
$$\frac{1}{T}\frac{\delta\Phi[\mathcal{G}]}{\delta\mathcal{G}(1,2)} = \Sigma(2,1) \;;\; \Phi_{\lambda=1}[\mathcal{G}] = \int_0^1 d\lambda \frac{1}{\lambda}\left\langle \lambda\hat{V}\right\rangle_\lambda \tag{44}$$

0.7 Lecture 6 (90 minutes) Lindhard function, TPSC and other approaches

Chapter 37 First steps with functional derivatives, Hartree-Fock and RPA
37.2 Hartree-Fock and RPA in space-time



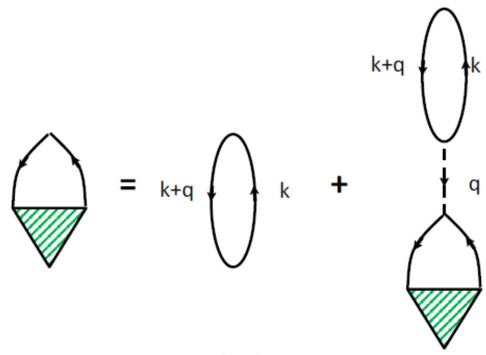
37.3 Hartre-Fock and RPA in momentum-Matsubara space



39.3 Density response in the non-interacting limit: Lindhard function

$$\chi_{nn}^{0R}(\mathbf{q},\omega) = -2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{f(\zeta_{\mathbf{k}}) - f(\zeta_{\mathbf{k+q}})}{\omega + i\eta + \zeta_{\mathbf{k}} - \zeta_{\mathbf{k+q}}} \tag{45}$$

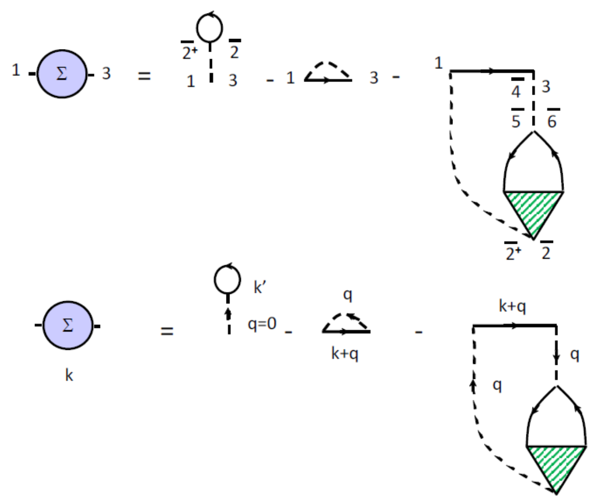
41.1.2 RPA



$$\chi_{nn}(q) = \frac{\chi^0_{nn}(q)}{1 + V_{\mathbf{q}}\chi^0_{nn}(q)} \tag{46}$$

Chapter 44 Second step of the approximation, GW curing Hartree-Fock

44.2 Self-energy and screening, GW



Chapter 56 Hubbard model in the footsteps of the electron gas

56.2 Response functions

$$U_{sp} = \frac{\delta\Sigma_\uparrow}{\delta\mathcal{G}_\downarrow} - \frac{\delta\Sigma_\uparrow}{\delta\mathcal{G}_\uparrow} \tag{47}$$

$$U_{ch} = \frac{\delta\Sigma_\uparrow}{\delta\mathcal{G}_\downarrow} + \frac{\delta\Sigma_\uparrow}{\delta\mathcal{G}_\uparrow} \tag{48}$$

56.3 Hartree-Fock and RPA

$$\chi_{sp}(q) = \frac{\chi_0(q)}{1 - \frac{1}{2}U\chi_0(q)} \tag{49}$$

$$\chi_{ch}(q) = \frac{\chi_0(q)}{1 + \frac{1}{2}U\chi_0(q)} \tag{50}$$

56.4 RPA and violation of the Pauli exclusion principle

$$\frac{T}{N}\sum_q \left(\frac{\chi_0(q)}{1 - \frac{1}{2}U\chi_0(q)} + \frac{\chi_0(q)}{1 + \frac{1}{2}U\chi_0(q)} \right) \neq 2n - n^2 \tag{51}$$

56.6 RPA, phase transitions and the Mermin-Wagner theorem

$$\mathbf{q}^2 \left\langle \phi_{\mathbf{q}} \phi_{-\mathbf{q}} \right\rangle = \frac{T}{2} \; ; \; \left\langle \phi^2 \right\rangle = \int_0^\infty \frac{d^2 q}{q^2} \frac{T}{2} = \infty \tag{52}$$

Chapter 57 The two-particle self-consistent approach TPSC

57.1 TPSC first step, spin and charge fluctuations

$$U_{sp} = U \frac{\langle n_\uparrow n_\downarrow \rangle}{\langle n_\uparrow \rangle \langle n_\downarrow \rangle} \; ; \; U_{ch} \text{ from Pauli} \tag{53}$$

57.2 An improved self-energy

$$\Sigma_\sigma^{(2)}(k) = U n_{-\sigma} + \frac{U}{8} \frac{T}{N} \sum_q \left[3 U_{sp} \chi_{sp}(q) + U_{ch} \chi_{ch}(q) \right] \mathcal{G}_\sigma^{(1)}(k+q) \tag{54}$$

57.3 An internal consistency check

$$\Sigma_\sigma \left(1, \overline{1}\right) \mathcal{G}_\sigma \left(\overline{1}, 1^+\right) \equiv \frac{1}{2} \mathrm{Tr} \left(\Sigma \mathcal{G} \right) = U \langle n_\uparrow n_\downarrow \rangle \tag{55}$$

$$\frac{1}{2} \mathrm{Tr} \left(\Sigma^{(2)} \mathcal{G}^{(1)} \right) = U \langle n_\uparrow n_\downarrow \rangle \tag{56}$$