4 1. $2^{\text{ème}}$ quant.
2. Motivé + déf. G^R
3. Repr. int. + T_t
4. Contours K.B. – K.S.
5. Matsubara
   1. Déf.
   2. Antipériodicité (F.D.)
   3. Prolong. analytique. $\mathcal{G}$ et G^R
   4. A repr. spectrale + règles prolongement.
   5. Sans interaction
   6. Fréquences de Matsubara 7. Asym

Résumé.

$$G^R(r,t;r't') \equiv -i \langle \{ \psi(r,t), \psi^+(r',t') \} \rangle \theta(t-t')$$

$$= -i \left[\langle \psi(t) \psi^+(t') \rangle + \langle \psi^+(t') \psi(t) \rangle \right] \theta(t-t')$$

$$= \left(G^>(t-t') - G^<(t-t') \right) \theta(t-t')$$

$$\rightarrow \langle U(0,t) \psi_s U(t,0) U(0,t') \psi_s^+ U(t',0) \rangle$$

$$e^{-iH_0 t} \quad e^{iH_0 t} \qquad e^{-iH_0 t} \quad e^{iH_0 t}$$

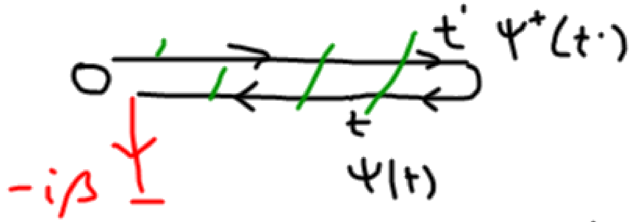
$$\hat{U}(0,t) = U(0,t) e^{-iH_0 t}$$

$$H = H_0 + V$$

$$i \frac{d\hat{U}(t,0)}{dt} = \hat{V}(t) \hat{U}(t,0)$$

$$\hat{U}(t,0) = T_t \, e^{-i \int_0^t dt' \hat{V}(t')}$$

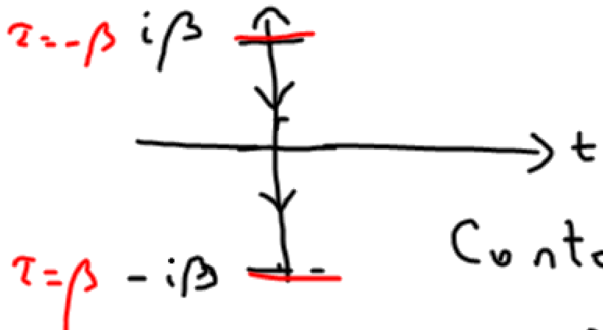
$$\langle\, \hat{U}(0,t)\, \hat{\Psi}(t)\, \hat{U}(t,t')\, \hat{\Psi}^{+}(t')\, \hat{U}(t',0) \,\rangle$$



$$e^{-\beta H} = e^{-i(-i\beta)H}$$

$$U(-i\beta, 0)$$

4.5 $\mathcal{G}$ + relation à G^R



$$\boxed{\tau \equiv it}$$

Contour Matsubara

$$\mathcal{G}(r,r';\tau) \equiv -\langle T_\tau\left[\psi(\tau)\,\psi^\dagger(0)\right]\rangle$$

$$= -\langle \psi(\tau)\psi^\dagger(0)\rangle\,\theta(\tau) + \langle \psi^\dagger(0)\psi(\tau)\rangle\,\theta(-\tau)$$

$$\langle O\rangle = \frac{\mathrm{Tr}\left[e^{-\beta(H-\mu N)}\,O\right]}{e^{-\beta\Omega}} \qquad e^{-\beta\Omega} = \mathrm{Tr}\left[e^{-\beta(H-\mu N)}\right]$$

$$\begin{cases} \Psi(\tau) \equiv e^{\tau(H-\mu N)} \Psi_s e^{-\tau(H-\mu N)} \\ \tau = it \\ \Psi^+(\tau) = e^{\tau(H-\mu N)} \Psi_s^+ e^{-\tau(H-\mu N)} \end{cases}$$

$$\boxed{K \equiv H - \mu N}$$

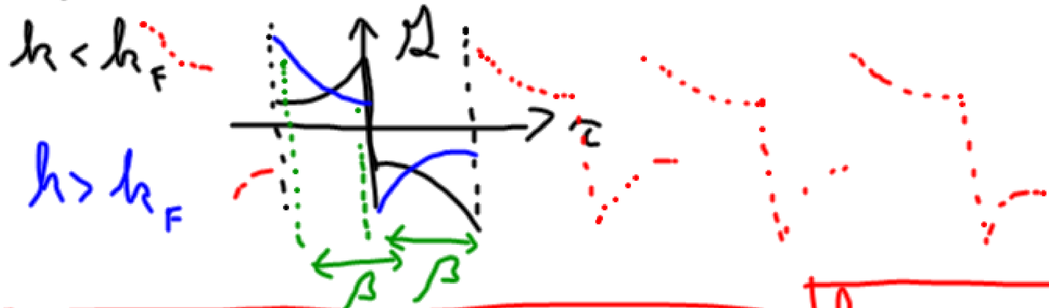
$$\tau = it$$

4.5.2 Antipériodicité :

$$\mathcal{G}(\tau) = -\mathcal{G}(\tau+\beta) \quad \text{si } \tau < 0$$

$$\mathcal{G}(\tau-\beta) = -\mathcal{G}(\tau) \quad \text{si } \tau > 0$$

e.g. cas sans interaction



$$\boxed{\mathcal{G}(\tau) = T\sum_{n=-\infty}^{\infty} e^{-ik_n\tau}\,\mathcal{G}(ik_n)} \qquad \boxed{k_n = (2n+1)\pi T}$$

$$\boxed{\mathcal{G}(ik_n) = \int_0^\beta d\tau\, e^{ik_n\tau}\,\mathcal{G}(\tau)}$$

Preuve d'antipériodicité

Si $\tau < 0 \Rightarrow \mathcal{G}(\tau) = \langle \psi^+(0)\psi(\tau) \rangle$

$$= e^{\beta\Omega}\, \mathrm{Tr}\left[e^{-\beta K}\psi^+ e^{\tau K}\psi e^{-\tau K} \right]$$

$$= e^{\beta\Omega}\, \mathrm{Tr}\left[e^{-\beta K} \underbrace{e^{+\beta K} e^{\tau K}} \psi \underbrace{e^{-\tau K} e^{-\beta K}} \psi^+ \right]$$

$$= \langle \psi(\tau+\beta)\psi^+(0) \rangle$$

4.5.3 Relation G^R et $\mathcal{G}$; A poids spectral

$$G^R(t) = -i \langle \{ \psi(t), \psi^+(t') \} \rangle \, \Theta(t-t')$$

$$G^R(t) = -i A(t) \, \theta(t)$$

$$A(t) \equiv \langle \{ \psi(t), \psi^+(0) \} \rangle$$

$$G^R(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(\omega')}{\omega + i\eta - \omega'}$$

$$\chi^R_{ij}(t) = 2i \chi''_{ij}(t) \, \theta(t)$$

$$\chi''_{ij}(t) = \langle [O_i(t), O_j(0)] \rangle$$

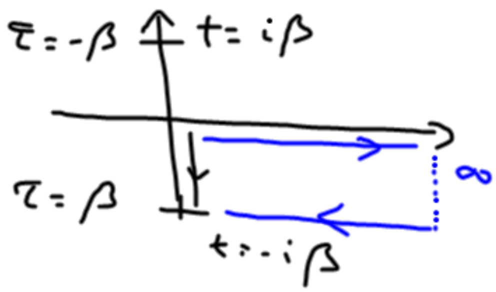
$$\chi^R_{ij}(\omega) = \int \frac{d\omega'}{\pi} \frac{\chi''_{ij}(\omega')}{\omega' - \omega - i\eta}$$

$$A(0) = \delta(r - r') = \int \frac{d\omega'}{2\pi} A(\omega')$$

$$\boxed{\mathcal{G}(ik_n) = \int \frac{d\omega'}{2\pi} \frac{A(\omega')}{ik_n - \omega'}}$$

À prouver

Preuve: $\mathcal{G}(ik_n) = \int_0^\beta d\tau\,(-)\langle \psi(\tau)\psi^+(0)\rangle\, e^{ik_n\tau}$



Soit $\mathrm{Re}\, k_n > 0$ alors on peut prendre $t > 0$ dans $\tau = it$ et l'exp. est décroissante à $t = +\infty$

$$\int_0^\infty d(it)\,(-)\langle \psi_H(t)\psi_H^+(0)\rangle\, e^{ik_n(it)}$$

$$+\int_\infty^0 d(it)\,(-)\,\langle \psi_H(t-i\beta)\psi_H^+(0)\rangle\, e^{ik_n i(t-i\beta)}$$

à prouver

$$e^{ik_n i(-i)\beta} = e^{ik_n\beta} = e^{i(2n+1)\pi \frac{T}{T}} = -1$$

$$\int_0^\infty d(it)\,(-)\left[\langle \psi_H(t)\psi_H^+(0)\rangle + \langle \psi_H^+(0)\psi_H(t)\rangle\right] e^{ik_n it}$$

$$= \int_0^\infty dt\left(-i\,\langle\{\psi_H(t), \psi_H^+(0)\}\rangle\right) e^{ik_n(it)}$$

$$= \int_{-\infty}^\infty dt\,\Big(\underbrace{-i\,\langle\{\ \}\rangle\,\theta(t)}_{G^R(t)}\Big)\, e^{ik_n(it)}$$

$$ik_n \to \omega + i\eta \qquad \eta > 0 \quad \mathrm{Re}\, k_n > 0$$

Preuve:

$$\langle \psi_H(t - i\beta)\, \psi^{\dagger}_H(0) \rangle = \langle \psi^{\dagger}_H(0)\, \psi_H(t) \rangle$$

$$\mathrm{Tr}\left[e^{-\beta K}\, e^{(it+\beta)K}\, \psi_H\, e^{(-it-\beta)K}\, \psi^{\dagger}_H(0) \right]\rangle$$

$$= \mathrm{Tr}\left[e^{-\beta K}\, \psi^{\dagger}_H(0)\, e^{itK}\, \psi_H\, e^{-itK} \right]$$

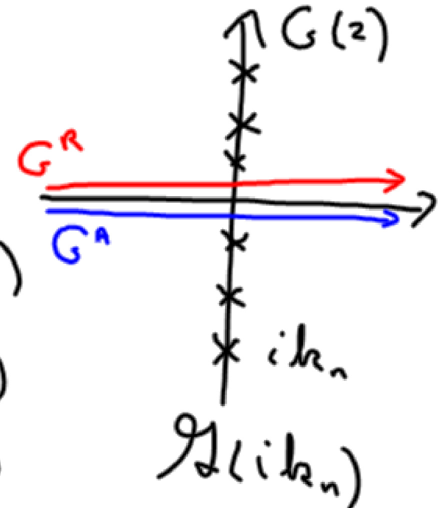
4.5.4 Poids spectral A et règles de prolongement analytique

$$\boxed{G(z) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(\omega')}{z-\omega'}}$$

$z = ik_n \Rightarrow G(z) = \mathcal{G}(ik_n)$

$z = \omega + i\eta \Rightarrow G(z) = G^R(\omega)$

$z = \omega - i\eta \Rightarrow G(z) = G^A(\omega)$



$$G^R = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(\omega')}{\omega + i\eta - \omega'}$$

Dimon

$$G(z)\left(1+\left(e^{\beta z}+1\right)\right)$$

Si $G(z)$ est analytique dans le demi-plan supérieur

Si $G(z) = \mathcal{G}(ik_n)$ $\quad ik_n$ Matsubara

Si $\lim\limits_{z \to \infty} G(z)\, z =$ constante.

$\Rightarrow$ Prolongement est unique.

$$G^R(\omega) = \lim_{ik_n \to \omega + i\eta} \mathcal{G}(ik_n)$$

4.5.5. Cas sans interaction

$$\boxed{\mathcal{G}(k,\tau-\tau') = -\langle T_\tau\, c_k(\tau)\, c_k^+(0)\rangle}$$

Démonstration à partir de l'espace r

$$\mathcal{G}(r,r';\tau-\tau') = -\langle T_\tau\, \psi(r,\tau)\psi^+(r',\tau')\rangle$$

$$= -\left\langle T_\tau \sum_k \sum_{k'} \langle r|k\rangle\, c_k(\tau)\, c^+_{k'}(\tau')\, \langle k'|r'\rangle \right\rangle$$

$$\frac{e^{+ikr}}{\sqrt{V}} \qquad\qquad \frac{e^{-ik'r'}}{\sqrt{V}}$$

$$e^{ikr-ik'r'} = e^{i\left(\frac{k+k'}{2}\right)(r-r') + i(k-k')\left(\frac{r+r'}{2}\right)}$$

$$\frac{1}{V}\int d\left(\frac{r+r'}{2}\right) e^{i(k-k')\left(\frac{r+r'}{2}\right)} = \frac{1}{V}(2\pi)^3\,\delta(k-k')$$

$$\mathcal{G}(k, ik_n) = \int \frac{d\omega'}{2\pi} \frac{A(k,\omega')}{ik_n - \omega'} = \frac{1}{ik_n - \xi_k}$$

$$A(k,\omega') = \int dt\, e^{+i\omega' t} \langle \{ c_k(t), c_k^\dagger(0) \} \rangle$$

$$\boxed{c_k(t) = e^{-i(\epsilon_k - \mu)t} c_k}$$

$$K = H - \mu N$$
$$= \sum_k (\epsilon_k - \mu) c_k^\dagger c_k$$

$$\boxed{\xi_k = \epsilon_k - \mu}$$

$$A(k,\omega') = \int dt\, e^{+i\omega' t} e^{-i\xi_k t} \langle \{ c_k, c_k^\dagger \} \rangle$$
$$= 2\pi\, \delta(\omega' - \xi_k)$$

$$H = -t \sum_{\substack{\langle ij \rangle \\ \sigma \\ n}} c^{\dagger}_{i\sigma,n} \, c_{j\sigma,n}$$

$$\Psi_{\sigma}(r) = \sum_{i,n} \langle r | i,n \rangle \, c_{in,\sigma}$$

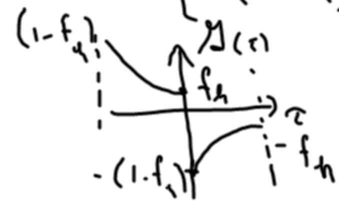
Calcul direct de $\mathcal{G}(k,\tau)$

$$\mathcal{G}(k,\tau) = -\langle c_k(\tau)\, c_k^{+}(0)\rangle\, \theta(\tau) + \langle c_k^{+}(0)\, c_k(\tau)\rangle\, \theta(-\tau)$$

$$= e^{-\xi_k \tau}\left[-\langle c_k c_k^{+}\rangle \theta(\tau) + \langle c_k^{+} c_k\rangle \theta(-\tau)\right]$$

$$\begin{cases} c_k(\tau) = e^{-\xi_k \tau} \\ \dfrac{\partial}{\partial \tau}\left(e^{\tau K} c_k e^{-\tau K}\right) = \left[K, c_k(\tau)\right] = -\xi_k c_k(\tau) \end{cases}$$

$$= e^{-\xi_k \tau}\left[-(1-f_k)\theta(\tau) + f_k\, \theta(-\tau)\right]$$



$k > k_F$

$$\mathcal{G}(ik_n) = \int_0^{\beta} e^{ik_n \tau}\, \mathcal{G}(\tau)\, d\tau$$

$$= \int_0^{\beta} d\tau\, e^{ik_n \tau} e^{-\xi_k \tau}(f_k - 1)$$

$$= \left.\frac{e^{ik_n\tau - \xi_k \tau}}{ik_n - \xi_k}\right|_0^{\beta} (f_k - 1)$$

$$= \frac{-e^{-\xi_k \beta} - 1}{ik_n - \xi_k}\left(-\frac{1}{1 + e^{-\beta \xi_k}}\right)$$

$$= \frac{1}{ik_n - \xi_k}$$

$$\langle j \rangle_{h.e.} = \langle j \rangle_e + \delta \langle j \rangle^{(1)} + \delta \langle j \rangle^{(2)}$$

$$\langle j\, j \rangle_e A \qquad \langle j\, j\, j \rangle_e A^2$$