CH. 1 : ELEMENTS OF TOPOLOGICAL BAND THEORY

[Ref: P. Bruno,
arXiv: 050627 (2005)]

M. Berry (1983):

A quantum system adiabatically transported around a closed circuit acquires, besides the familiar dynamical phase, a phase that depends only on the geometry of the circuit.

"Berry phase": central concept in quantum mechanics, with applications ranging from chemistry to condensed matter physics to quantum information.

1 Parallel transport in geometry

Consider a surface S, and a vector tangent to it.
We want to transport the vector on the surface:

(i) vector is always tangent to S

(ii) do not rotate vector around the axis normal to the surface.

(i) + (ii) : "parallel transport".

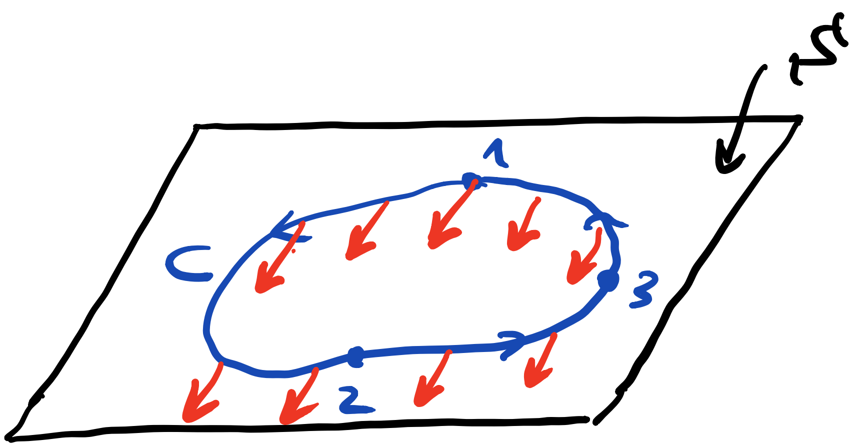
Consider a closed path C :

$$1 \to 2 \to 3 \to 1$$

Two scenarios :

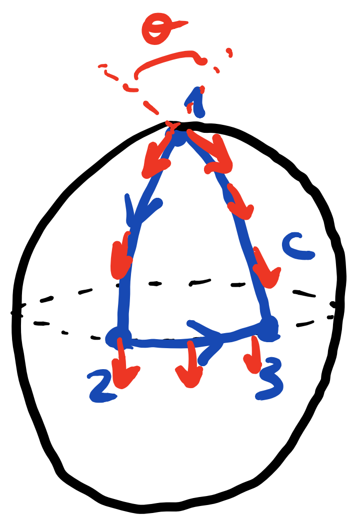
(1) $\mathcal{S}$ is flat

$\Rightarrow$ the vector that is transported always points in the same direction.



(2) S^2 is curved

$\Rightarrow$ the vector changes direction as it moves along C.



The vector's initial direction differs from the final one by an angle $\theta(C)$

"holonomy"

geometric counterpart of the Berry phase.

A more quantitative discussions

Transported vector: $\hat{e}_1$

$$\hat{e}_1 \cdot \hat{e}_1 = 1$$

Loop: $C \equiv \{ \vec{r}(t) \mid \underbrace{t}_{\text{time}} : 0 \to T \}$

$$\vec{r}(0) = \vec{r}(T)$$

$\hat{n}$ = unit vector normal to S.

$= \hat{n}(\vec{r})$

$\hat{e}_1 \cdot \hat{n} = 0$ for all $\vec{r} \in S$

Another tangent vector:

$$\hat{e}_2 = \hat{n} \times \hat{e}_1$$

$$(\hat{e}_2 \cdot \hat{n} = 0)$$

$(\hat{e}_1, \hat{e}_2, \hat{n})$ = orthonormal reference system.

Equation of motion for $\hat{e}_1$ as it is transported along C ?

Impose that $\hat{e}_1 \cdot \hat{e}_1 = 1$ for all $\vec{r} \in C$

$$\Rightarrow \underbrace{\frac{d}{dt}(\hat{e}_1 \cdot \hat{e}_1)}_{2\hat{e}_1 \cdot \frac{d\hat{e}_1}{dt}} = 0$$

$$\Rightarrow \quad \frac{d\hat{e}_1}{dt} \perp \hat{e}_1$$

$$(\dot{\hat{e}}_1 \perp \hat{e}_1)$$

$$\boxed{\dot{\hat{e}}_1 = \underbrace{\vec{\omega}} \times \hat{e}_1} \leftarrow$$

angular frequency of rotation (TBD)

similarly,

$$\boxed{\dot{\hat{e}}_2 = \vec{\omega} \times \hat{e}_2} \leftarrow$$

Need :

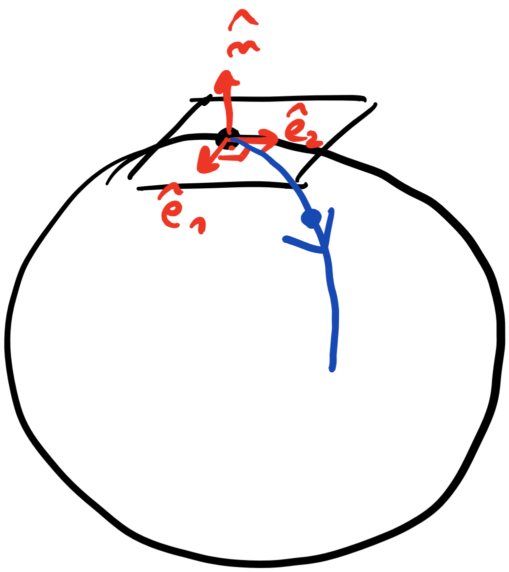
(i) $\hat{e}_i \cdot \hat{n} = 0$ $(i = 1, 2)$

(ii) $\underline{\vec{\omega} \cdot \hat{n}} = 0$ ✓

(parallel transport)

$$\Longrightarrow \quad \boxed{\vec{\omega} = \hat{n} \times \dot{\hat{n}}}$$

they're satisfied if



In sum,

$$\boxed{\dot{\hat{e}}_i} = (\hat{n} \times \dot{\hat{n}}) \times \hat{e}_i$$

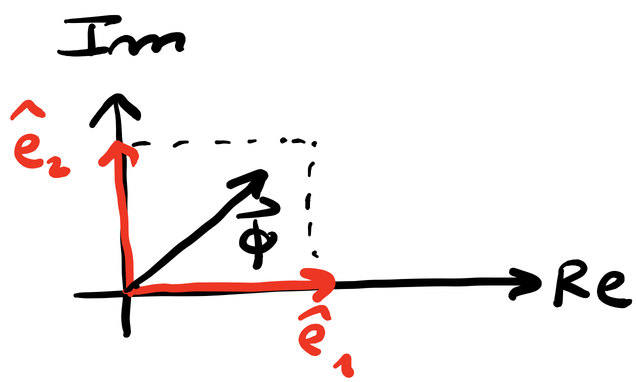
$$= \boxed{-(\hat{e}_i \cdot \dot{\hat{n}})\,\hat{n}} \qquad (1)$$

standard vector relation

"law of parallel transport"

Let's reformulate this (in anticipation to its generalisation to quantum case):

$$\boxed{\vec{\phi} \equiv \frac{1}{\sqrt{2}}\left(\hat{e}_1 + i\,\hat{e}_2\right)}$$



$$\vec{\phi}^{\,*} \cdot \vec{\phi} = 1$$

The law of parallel transport (1)

implies

$$\boxed{\vec{\phi}^{\,*} \cdot \dot{\vec{\phi}} = 0} \qquad (2)$$

To express the rotation of $\hat{e}_1$ and $\hat{e}_2$ as they move along C, we choose a fixed local orthonormal frame on S^1:

$$\Big(\underbrace{\hat{t}_1 \, , \, \hat{t}_2}_{\text{tangent to } S^1} \, , \, \underbrace{\hat{n}}_{\perp \text{ to } S^1} \Big)$$

$$\hat{t}_i = \hat{t}_i \, (\vec{r})$$

Impose:

$$\left\{ \begin{array}{l} \hat{t}_1 \text{ is continuously differentiable} \\ \text{w.r.t. } \vec{r} \\ \hat{t}_1 \text{ is single-valued.} \end{array} \right.$$

notice that $(\hat{e}_1, \hat{e}_2, \hat{n})$ is not single-valued at point 1 if $\theta \neq 0$.

$$\hat{t}_2 = \hat{n} \times \hat{t}_1$$

example:

$$(\hat{t}_1, \hat{t}_2, \hat{n}) = \underbrace{(\hat{\theta}, \hat{\phi}, \hat{r})}_{\text{unit vectors in spherical coordinates.}}$$

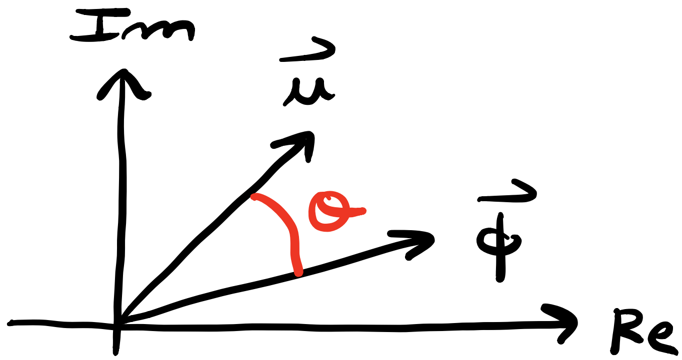
Now define

$$\boxed{\vec{u} \equiv \frac{1}{\sqrt{2}} \left(\hat{t}_1 + i \hat{t}_2 \right)}$$

$$\vec{u}^* \cdot \vec{u} = 1$$

Relation between $(\hat{e}_1, \hat{e}_2, \hat{n})$ and $(\hat{t}_1, \hat{t}_2, \hat{n})$:

$$\boxed{\vec{\phi}(t) = e^{-i\theta(t)} \, \vec{u}(t)} \qquad (3)$$



θ : angle by which we need to rotate $(\hat{t}_1, \hat{t}_2)$ to make them coincide with $(\hat{e}_1, \hat{e}_2)$.

Equation for θ ?

From (2) :

$$0 = \vec{\phi}^* \cdot \dot{\vec{\phi}}$$

$$\overset{(3)}{=} -i\dot{\theta} + \vec{u}^* \cdot \dot{\vec{u}}$$

$$\Rightarrow \boxed{\dot{\theta} = -i\, \vec{u}^* \cdot \dot{\vec{u}}}$$

θ must be real

$\Rightarrow$ $\vec{u}^* \cdot \dot{\vec{u}}$ is pure imaginary.

$$\dot{\theta} = \mathrm{Im}\left(\vec{u}^* \cdot \dot{\vec{u}}\right)$$

$$\Rightarrow \boxed{\theta = \mathrm{Im} \int_0^T \vec{u}^* \cdot \dot{\vec{u}}\, dt}$$

$\theta \equiv 0$ at $t = 0$.

Alternatively,

$$\boxed{\theta = \mathrm{Im} \oint_c \vec{u}^* \cdot d\vec{u}}$$

$$\dot{\vec{u}}\, dt = \frac{d\vec{u}}{dt}\, dt = d\vec{u}$$