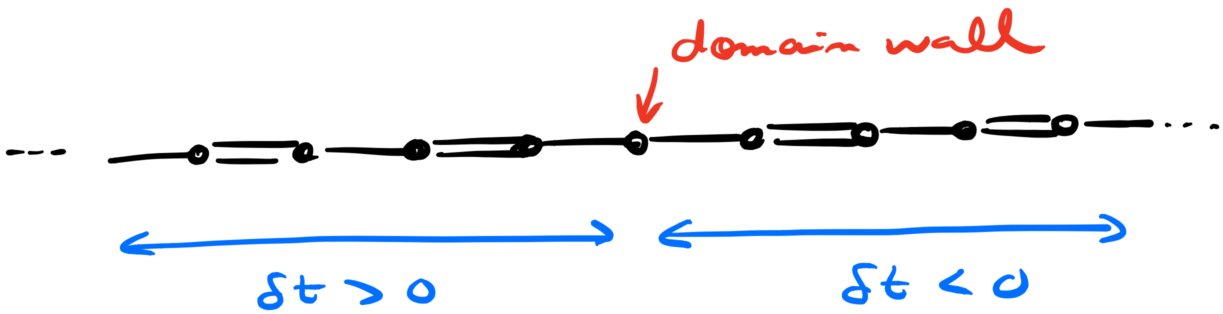
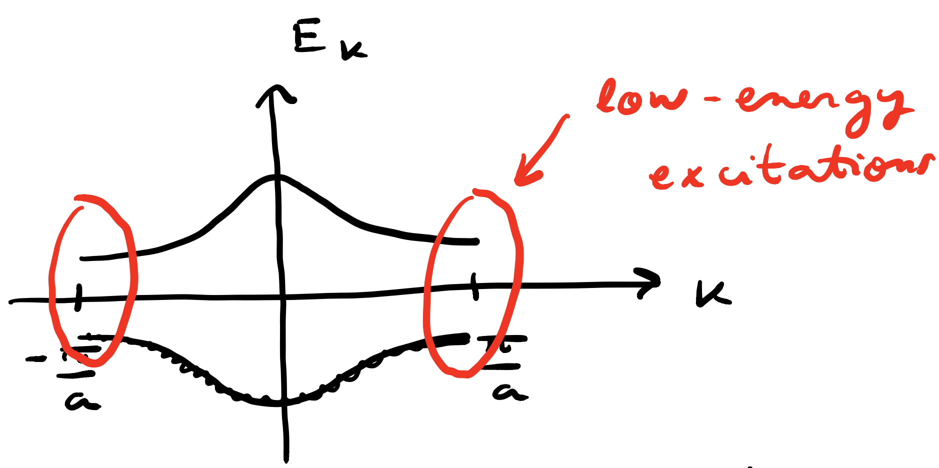
3.3 Zero-energy edge modes



* Low-energy effective theory



$$t \gg |\delta t|$$

Take $\quad k = \frac{\pi}{a} - q$

where $\quad qa \ll 1$

$$h(k) = \vec{d}(k) \cdot \vec{\sigma}$$

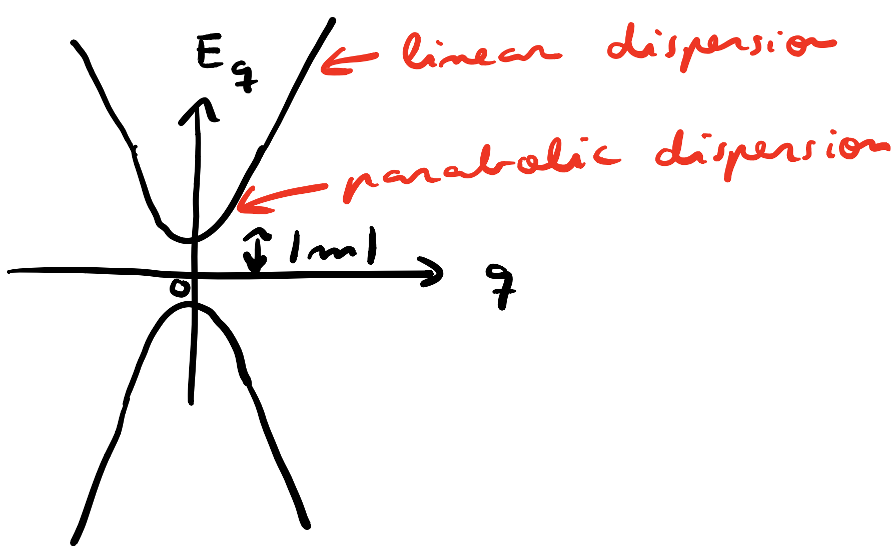
Then,

$$d_x(k) \approx 2\,\delta t$$
$$d_y(k) \approx a t q$$
$$d_z(k) = 0$$

$$\Rightarrow \boxed{h_{eff}(q) = m\,\sigma^x + v q\,\sigma^y}$$

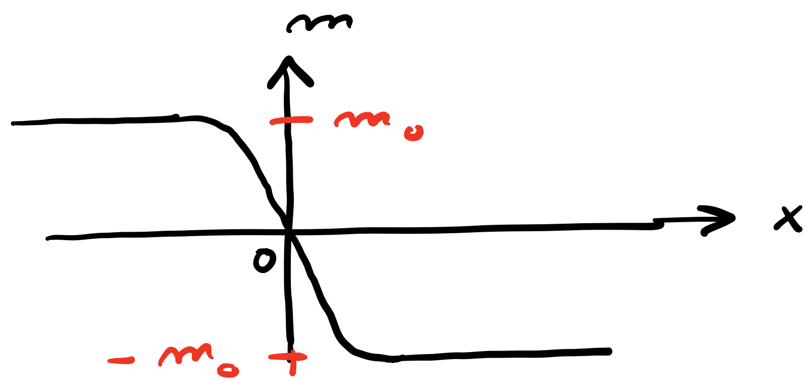
where
$$\begin{cases} m \equiv 2\,\delta t = \text{Dirac mass} \\ v \equiv a t = \text{Dirac velocity} \end{cases}$$

1D Dirac Hamiltonian



$$E_q = \pm\sqrt{v^2q^2+m^2}$$

Domain wall: sign change of m.



$$q \to -i\frac{\partial}{\partial x}$$

$$h_{eff} = m(x)\,\sigma^x - i\,v\,\sigma^y\frac{\partial}{\partial x}$$

This has a zero-energy solution.

"Jackiw-Rebbi zero mode"

(PRD 13, 3398 (1976))

Proof :

zero - energy eigenstate

Try $h_{eff}\, \psi = 0$

$$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$$

$$\begin{pmatrix} 0 & m - v\partial_x \\ m + v\partial_x & 0 \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} (m - v\partial_x)\,\psi_\downarrow = 0 \\ (m + v\partial_x)\,\psi_\uparrow = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \psi_\downarrow(x) = A\, e^{\frac{1}{v}\int_0^x dx'\, m(x')} \\ \psi_\uparrow(x) = B\, e^{-\frac{1}{v}\int_0^x dx'\, m(x')} \end{cases}$$

A, B : integration constants.

In our case,

$$m(x>0) < 0$$

$$m(x<0) > 0$$

$\Longrightarrow$

$$e^{-\frac{1}{v}\int_0^x dx'\, m(x')}$$

blows up for $x \to \pm\infty$,

while

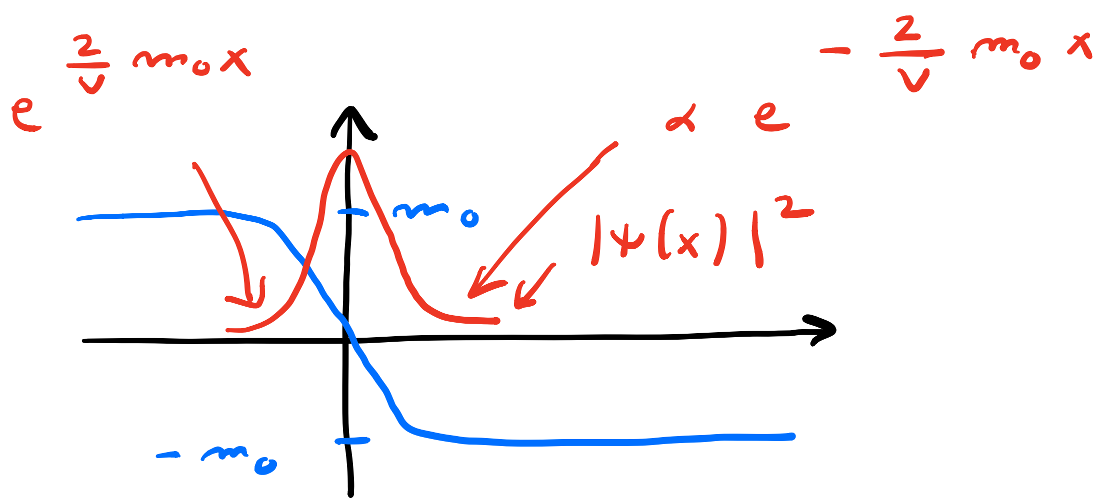
$$e^{\frac{1}{v}\int_0^x dx'\, m(x')}$$

does not.

The only physical solution

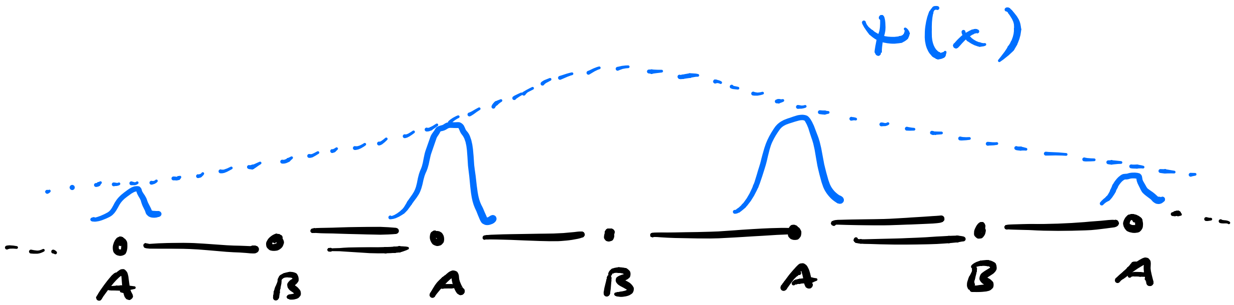
is $\qquad B = 0$:

$$\psi(x) \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\frac{1}{v}\int_0^x dx'\, m(x')}$$



zero mode is exponentially localised at the domain wall, w/ a decay length $\sim \frac{v}{m_0}$

Moreover, $\psi(x) \neq 0$ only for A sites.

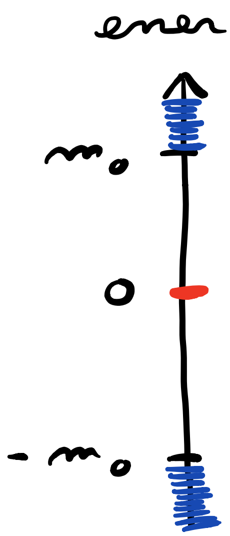


If we reverse the sign of $m(x)$ everywhere, then the zero mode is

$$\psi(x) \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{1}{v}\int_0^x dx'\, m(x')}$$

localised on B sites.

* Overall energy spectrum:



— : "bulk states" (wave function is delocalised)

— : JR mode (bound state)

Energy of JR mode remains pinned to zero even when we "deform" the Hamiltonian, provided that $d_z = 0$

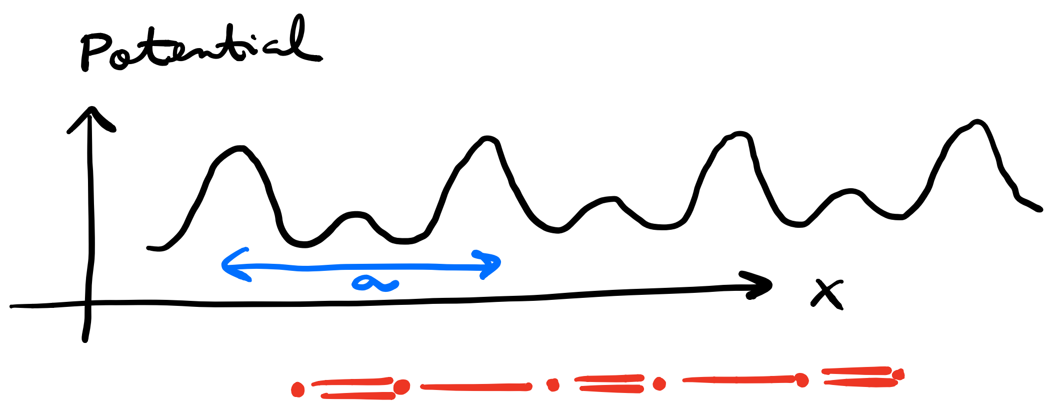
"zero-mode is protected by chiral symmetry"

3.4 Experimental realisation

M. Atala et al.,

Nature Physics 9, 795 (2013)

Cold atoms in a 1D optical lattice



Prepare an atom in an initial state

$$|\downarrow, k = 0\rangle$$

spin of atom

$$\xrightarrow[\text{MW pulse}]{} \frac{1}{\sqrt{2}}\Big[\,|\uparrow, k=0\rangle + |\downarrow, k=0\rangle\,\Big]$$

Apply a magnetic field gradient

$\Rightarrow$ $|\uparrow\rangle$ and $|\downarrow\rangle$ suffer an opposite force.

Time-evolution

$$\frac{1}{\sqrt{2}}\Big[\,|\uparrow, k\rangle + e^{i\delta\varphi}|\downarrow, -k\rangle$$

k increases w/ time.

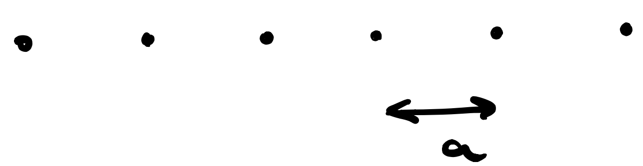
when $k = \frac{\pi}{a}$,

$\delta\varphi$ = Berry phase + dynamical phase.

Need to subtract the dynamical phase.

Finally, do an interference measurement ("Ramsey" interferometry) to extract Berry phase.

4 Kitaev's model



1D lattice.

Spinless electrons.

Superconductivity.

Grand-canonical Hamiltonian:

$$\mathcal{H} = -t \sum_j \left(c_j^+ c_{j+1} + h.c. \right)$$
$$- \mu \sum_j c_j^+ c_j$$
$$+ \Delta \sum_j \left(c_j c_{j+1} + h.c. \right)$$

t = nearest-neighbor hopping

μ = chemical potential

Δ = superconducting pairing potential.

Assume $t \in \mathbb{R}, > 0$

$\Delta \in \mathbb{R}$

Fourier transform from j to k.

$$\mathcal{H} = \sum_k \left(c_k^+ \quad c_{-k} \right) h(k) \begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix}$$

where

$$h(k) = \vec{d}(k) \cdot \vec{\sigma}$$

electron - hole

pseudospin

$\sigma^z \uparrow$: electron

$\sigma^z \downarrow$: hole.

$$d_x(k) = \Delta \sin(ka)$$

$$d_y(k) = 0$$

$$d_z(k) = \underbrace{-2t\cos(ka)}_{\text{usual tight binding dispersion}} - \mu$$

in 1D

$d_x \neq 0 \Rightarrow$ eigenstates are linear combinations of electrons and holes.

(essence of superconductivity)

$d_x(k) = -d_x(-k)$

$\Rightarrow$ odd-parity superconductivity.

Consequence of Pauli principle and spinless electrons.

$$\sum_k d_x(k)\, c_k^+ c_{-k}^+ \quad (+h.c.)$$

$$\underset{k \to -k}{=} \sum_k d_x(-k)\, c_{-k}^+ c_k^+ \quad (+h.c.)$$

$$= - \sum_k d_x(-k)\, c_k^+ c_{-k}^+ \quad (+h.c.)$$

$$c_k^+ c_{-k}^+ = - c_{-k}^+ c_k^+ \quad \text{(Pauli)}$$

$$\Rightarrow \quad d_x(k) = - d_x(-k)$$

Formally, $h(k)$ looks very similar to that of the SSH model $\Rightarrow$ we can repeat much of the analysis.