4) Kitaev's model

1D superconductor of spinless e^--s.

$$\mathcal{H} = \sum_k \left(c_k^+ \;\; c_{-k} \right) h(k) \begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix},$$

where

$$h(k) = \vec{d}(k) \cdot \underbrace{\vec{\sigma}}_{\text{electron-hole pseudospin}}$$

$$d_x(k) = \overset{\text{SC pairing}}{\Delta} \sin(ka) \quad \text{odd in } k$$

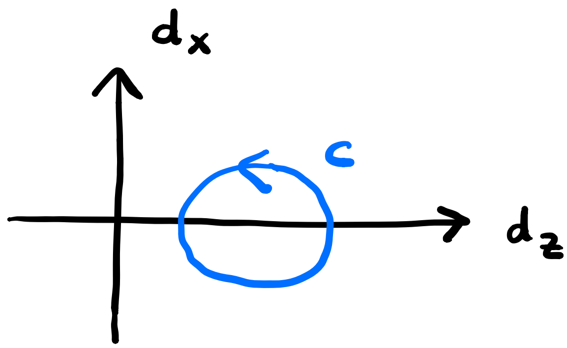
$$d_y(k) = 0 \qquad\qquad (t > 0)$$

$$d_z(k) = -2\underbrace{t}_{\text{nearest-neighbor hopping}} \cos(ka) - \underbrace{\mu}_{\text{chemical potential}}$$

$\{ h(k), \sigma_y \} = 0 \Rightarrow$ chiral symmetry

$\Rightarrow$ two topologically distinct phases:

(1) $|\mu| > 2t$

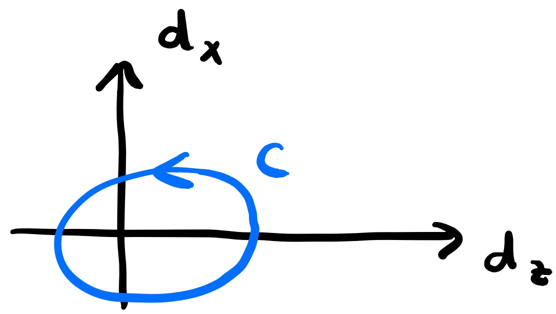


c = path of $\vec{d}$ as k is varied from $-\pi/a$ to $+\pi/a$.

$\gamma_\pm = 0$

"trivial superconductor"

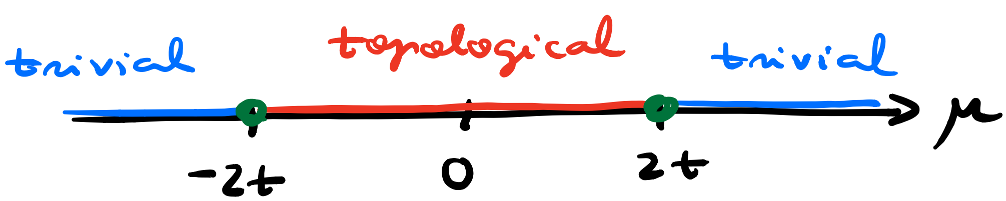
(2) $|\mu| < 2t$



$\gamma_\pm = \pi$

"topological SC"

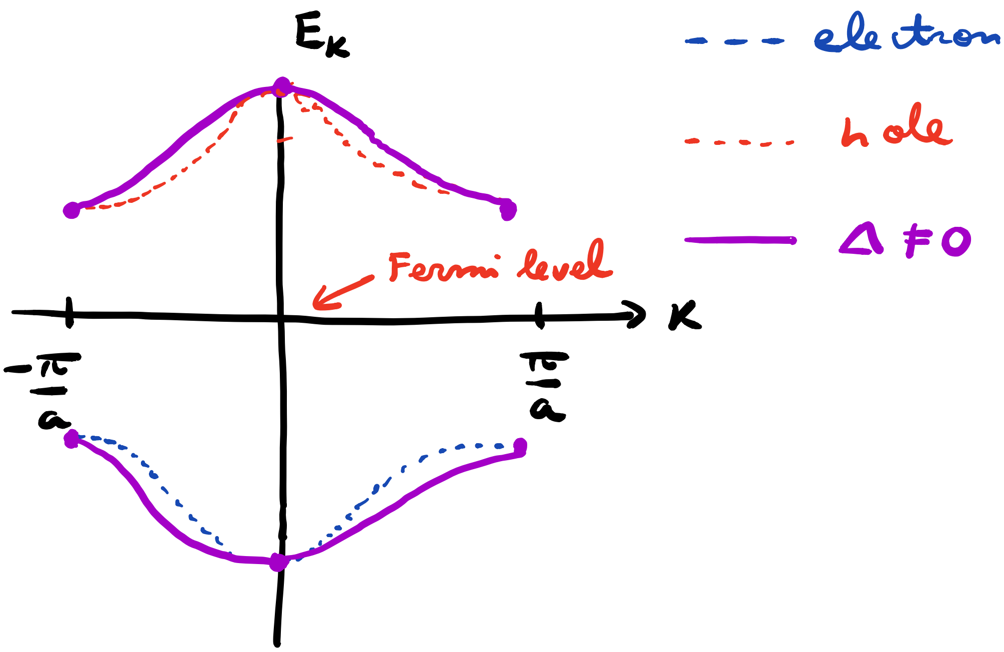
* Phase diagram :



• topological phase transition

* Band structures

(1) Trivial SC $(|\mu| > 2t)$



Normal state : insulator.

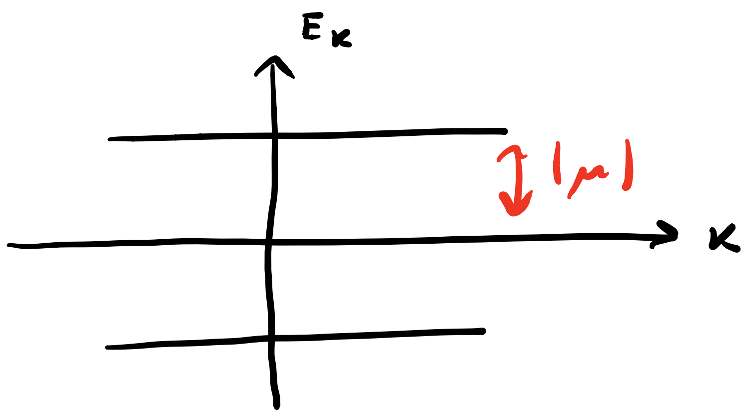
"strong pairing phase"

This band structure is smoothly connected to vacuum.

w/o closing gap.

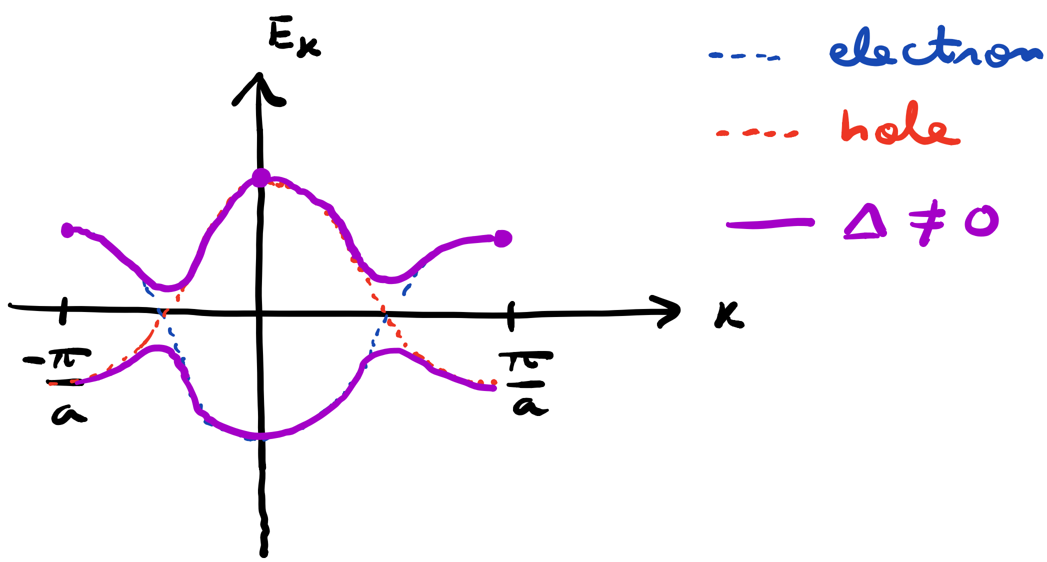
$t \to 0$

$\Delta \to 0$



This is another reason why this phase is called "trivial"

(2) Topological SC ($|\mu| < 2t$):



Superconductivity emerges from a metal

("weak pairing phase")

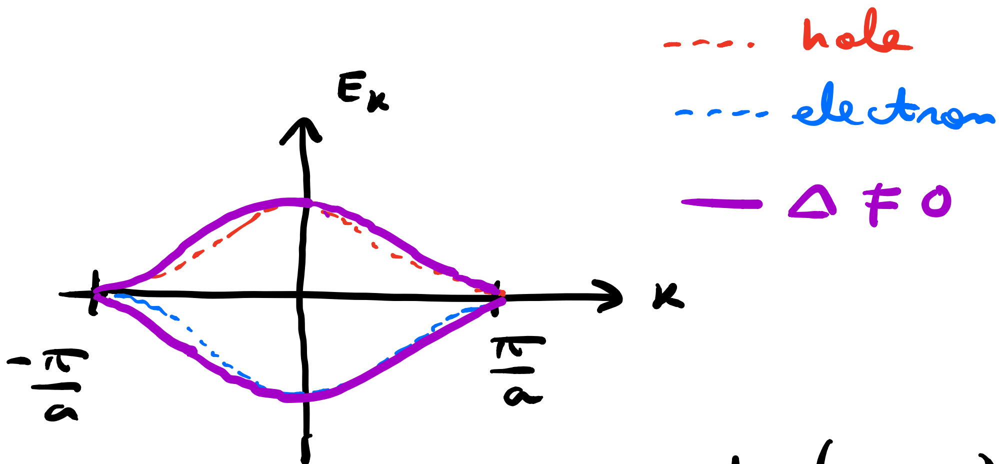
This phase is not smoothly connected to vacuum.

As t decreases, $2t$ will become equal to μ. At that point, the gap closes.

this is another reason why the phase is called "nontrivial"

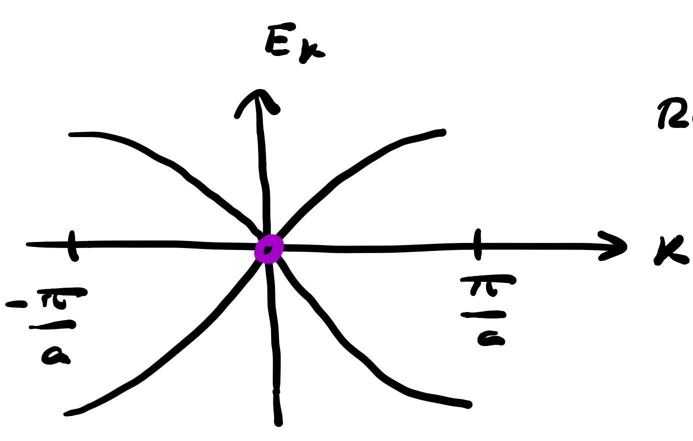
* Gap closing points : $\mu = \pm 2t$

(1) $\mu = 2t$



Recall : $d_x(\pm\frac{\pi}{a}) = 0$

(2) $\mu = -2t$



Recall : $d_x(0) = 0$.

4.1 Majorana zero modes

Consider $\mu \approx -2t$

Low-energy excitations occur at $k \simeq 0$.

$\Rightarrow$ expand $h(k)$ near $k=0$:

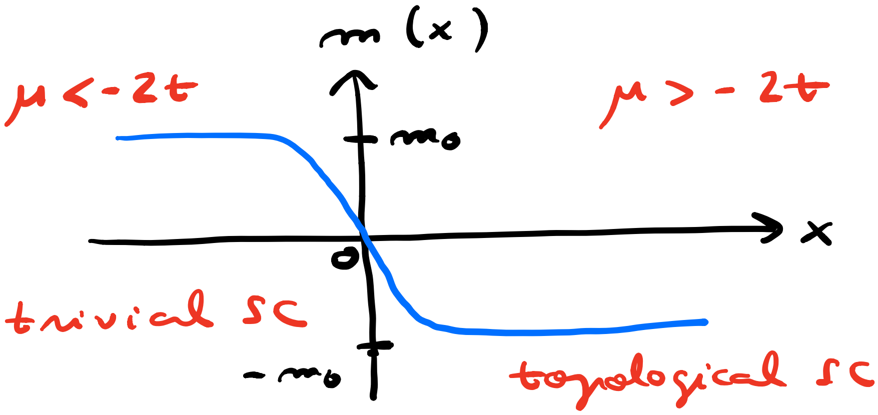
$$h(k) \approx m\,\sigma^z + v\,k\,\sigma^x$$

Massive 1D Dirac fermions.

$m \equiv$ Dirac mass $= -2t-\mu$

$v \equiv$ " velocity $= a\Delta$

Consider a domain wall:



Sign-change of Dirac mass

$\rightarrow$ localized zero-energy mode.

$$k \rightarrow -i\,\partial_x$$

$$h\,\Psi = 0$$

$$\begin{pmatrix} m & -iv\,\partial_x \\ -iv\,\partial_x & -m \end{pmatrix} \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Only one normalisable solution.

$$\psi(x) = \underbrace{e^{i\frac{\pi}{4}}}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{\frac{1}{v}\int_0^x dx' \, m(x')}$$

can always multiply a phase.

wave function $\longrightarrow$ operator

$\psi(x)$: operator that annihilates the zero mode.

$\psi^+(x)$: operator that creates the zero mode.

It turns out that $\psi^+(x) = \psi(x)$

$\Rightarrow$ "Majorana"

Proof:

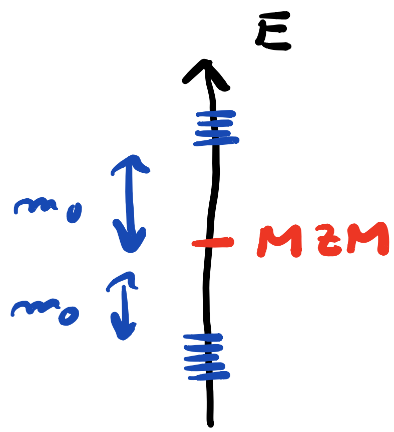
$$\psi(x) = e^{i\frac{\pi}{4}}(c - ic^+)\, e^{\frac{1}{v}\int \cdots}$$

$$\psi^+(x) = \underbrace{e^{-i\frac{\pi}{4}}(c^+ + ic)}_{\underbrace{e^{i\frac{\pi}{4}}e^{-i\frac{\pi}{2}}(c^+ + ic)}_{e^{i\frac{\pi}{4}}(-ic^+ + c)}}\, e^{\frac{1}{v}\int \cdots}$$

$$\Rightarrow \psi^+ = \psi$$

The localised zero mode is its own antiparticle

"Majorana zero mode" (MZM)



Let $|GS\rangle$ be the many-body ground state of the Kitaev model.

$$\mathcal{H}\,|GS\rangle = E_{GS}\,|GS\rangle$$

We showed that

$\psi^\dagger |GS\rangle$ is also an eigenstate of $\mathcal{H}$, with zero excitation energy

MZM creation operator

$$\mathcal{H}\left(\psi^+|GS\rangle\right) = E_{GS}\left(\psi^+|GS\rangle\right)$$

Now consider

$$[\mathcal{H}, \psi^+]\,|GS\rangle$$

$$= \mathcal{H}\left(\psi^+|GS\rangle\right) - \psi^+\left(\mathcal{H}|GS\rangle\right)$$

$$= E_{GS}\,\psi^+|GS\rangle - E_{GS}\,\psi^+|GS\rangle$$

$$= 0$$

$$\Rightarrow \boxed{[\mathcal{H}, \psi^+] = 0}$$

Change of notation:

write

$$\psi(x) = \Gamma\, e^{\frac{1}{v}\int_0^x dx'\, m(x')}$$

operator.

$$\Gamma = e^{i\frac{\pi}{4}}\left(c - i c^{+}\right)$$

Then,

- $\Gamma = \Gamma^{+}$
- $\{\Gamma, \Gamma^{+}\} = 2$
- $\{\Gamma, \Gamma\} = \{\Gamma^{+}, \Gamma^{+}\} = 2$

not a fermion.

for a fermion,

$$\{c, c\} = \{c^{+}, c^{+}\} = 0.$$

$$\{\Gamma, \Gamma\} = 2\,\Gamma^{2}$$

$$P^2 = e^{i\frac{\pi}{2}}\,(c - i c^+)(c - i c^+)$$

$$= i\,(\cancel{c}\,c^{\nearrow 0} - i c c^+ - i c^+ c - \cancel{c^+}\,c^{+\nearrow 0})$$

$$= c c^+ + c^+ c = \{c, c^+\}$$

$$= 1$$

4.2 Experimental realisation

* Ingredients:

(i) Nanowire (1D)

(ii) Conventional (s-wave) SC

(excitations are linear combinations of e^--s and holes)

(iii) Spin-orbit coupling in the nanowire. (→ p-wave SC)

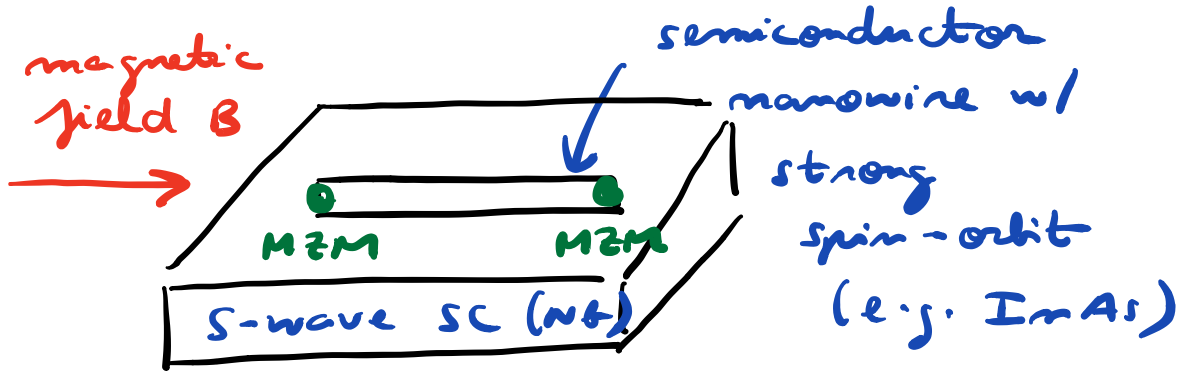
(iv) Zeeman field (spinless fermions)

Latest reviews:

Frolov et al., Nature Physics 16, 718 (2020)

Prada et al., Nature Rev. Physics 2, 575 (2020)

* A particular (popular) realisation:

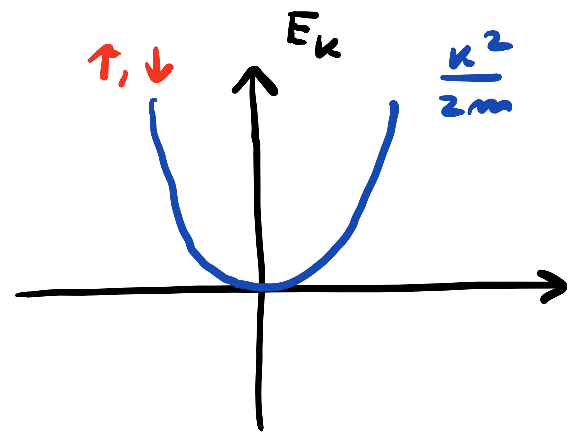


Hamiltonian for the nanowire:

$$\mathcal{H} = \sum_{k} \left(c^{\dagger}_{k\uparrow} , c^{\dagger}_{k\downarrow} \right) \left[\left(\frac{k^2}{2m} - \mu \right) \mathbb{1} + \lambda_{so} \, k \, \sigma^{y} + B \, \sigma^{x} \right] \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix}$$

$$+ \underbrace{\Delta_0}_{\text{proximity-induced pairing}} \sum_{k} \underbrace{\left(c^{\dagger}_{k\uparrow} \, c^{\dagger}_{-k\downarrow} + c_{-k\downarrow} \, c_{k\uparrow} \right)}_{\text{conventional s-wave pairing.}}$$

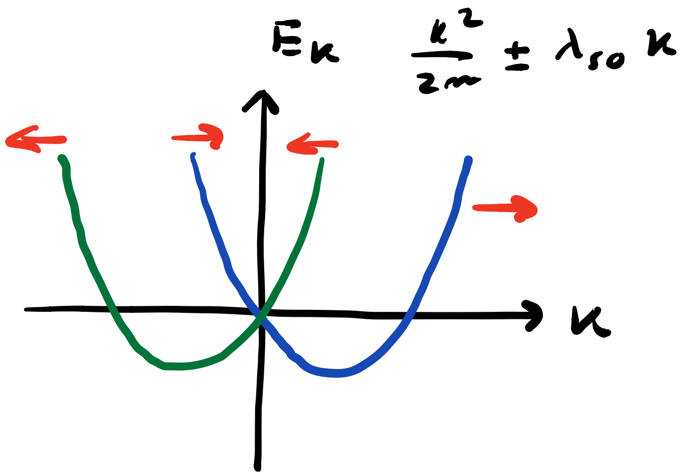
* Band structures in the normal state ($\Delta_0 = 0$)



$B = 0$

$\lambda_{so} = 0$

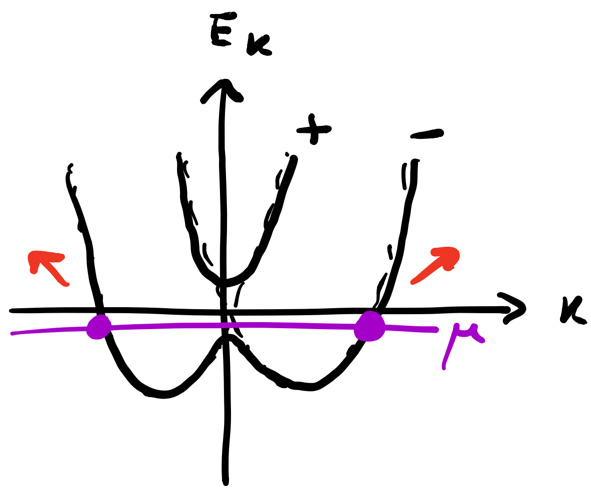
(only e^- bands)



$B = 0$

$\lambda_{so} \neq 0$

Spins along $\pm y$ direction.



$B \neq 0$

$\lambda_{so} \neq 0$

"$B \perp \lambda_{so}$"

way to engineer spinless fermions.

Eigenstates in the normal state:

$$|\psi_{k+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\varphi_k} \end{pmatrix}$$

$$|\psi_{k-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ e^{i\varphi_k} \end{pmatrix}$$

$$\tan \varphi_k = \frac{\lambda_{so} k}{B}$$

* Write $\mathcal{H}$ in the eigenstate basis:

$$\mathcal{H} = \sum_k \epsilon_{k-} c^{+}_{k-} c_{k-}$$

$$+ \sum_k \epsilon_{k+} c^{+}_{k+} c_{k+}$$

$$+ \frac{\Delta_0}{2} \sum_k e^{i\varphi_k} \left(c_{-k+} - c_{-k,-} \right) \times \left(c_{k+} + c_{k-} \right)$$

$$+ h.c.$$

$$\Delta_0 \sum \left(c_{-k\downarrow} c_{k\uparrow} + h.c. \right)$$

Need to write $c_{k\uparrow}$ and $c_{-k\downarrow}$ in the eigenstate basis.

$$|\psi_{k+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\varphi_k} \end{pmatrix}$$

$$|\psi_{k-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ e^{i\varphi_k} \end{pmatrix}$$

$$c_{k+} = \frac{1}{\sqrt{2}} \left(c_{k\uparrow} + e^{i\varphi_k} c_{k\downarrow} \right)$$

$$c_{k-} = \frac{1}{\sqrt{2}} \left(- c_{k\uparrow} + e^{+i\varphi_k} c_{k\downarrow} \right)$$

$$\begin{pmatrix} c_{k+} \\ c_{k-} \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\varphi_k} \\ -1 & e^{i\varphi_k} \end{pmatrix}}_{\text{unitary matrix}} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix}$$

Invert this :

$$\begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ e^{-i\varphi_k} & e^{-i\varphi_k} \end{pmatrix} \begin{pmatrix} c_{k+} \\ c_{k-} \end{pmatrix}$$

plug this into

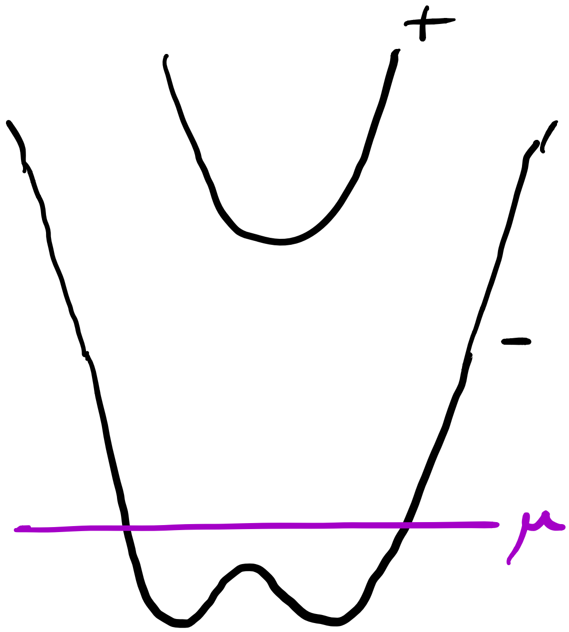
$$\Delta_0 \sum_k \left(c_{k\uparrow} c_{-k\downarrow} + h.c. \right)$$

This gives

$$\frac{\Delta_0}{2} \sum_k e^{i\varphi_k} \left(c_{-k,+} - c_{-k,-} \right) \left(c_{k+} + c_{k-} \right)$$

$$+ h.c.$$

* Low energy effective theory



+ band is far from Fermi level.

Essentially we can ignore the states from + band :

$$c_{k+} \to 0$$

$$\mathcal{H}_{eff} \approx \sum_{k} \epsilon_{k-} c^{+}_{k-} c_{k-}$$

$$+ \frac{\Delta_0}{2} \sum_{k} \Big(e^{i\varphi_k} c_{-k,-} c_{+k,-}$$

$$+ h.c \Big)$$

Let's check that the pairing term has odd parity.

$$\frac{\Delta_0}{2} \sum_{k} e^{i\varphi_k} c_{-k} c_{k}$$

$$= \frac{\Delta_0}{4} \sum_{k} e^{i\varphi_k} c_{-k} c_{k}$$

$$+ \frac{1}{2} \sum_{k} e^{i\varphi_k} c_{-k} c_{k}$$

$$= \frac{1}{2} \sum_k e^{i\varphi_k} c_{-k} c_k$$

$$+ \frac{1}{2} \sum_k e^{i\varphi_{-k}} \underbrace{c_k c_{-k}}_{-c_{-k} c_k}$$

$$= \frac{1}{2} \sum_k \left(e^{i\varphi_k} - e^{i\varphi_{-k}} \right) c_{-k} c_k$$

$$= i \sum_k \sin(\varphi_k) \, c_{-k} c_k$$

$\varphi_{-k} = -\varphi_k$

$$= i \sum_k \frac{\lambda_{so} k}{\sqrt{B^2 + (\lambda_{so} k)^2}} c_{-k} c_k$$

$$\sin \varphi_k = \frac{\tan \varphi_k}{\sqrt{1 + \tan^2 \varphi_k}} = \frac{\lambda_{so} k}{\sqrt{B^2 + (\lambda_{so} k)^2}}$$

Odd-parity pairing, proportional to spin-orbit coupling.

The amplitude of p-wave pairing is governed by the dimensionless ratio $\frac{\lambda_{so} k_F}{B}$.