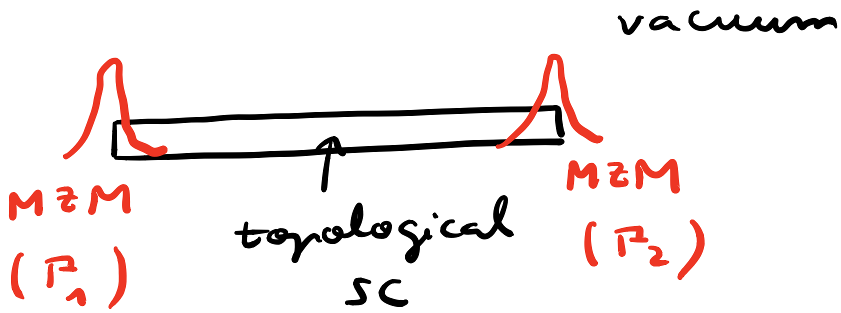
4.3 Application: topological quantum computation

* Majorana qubit



MZM operators: Γ_1 , Γ_2

$$[\mathcal{H}, \Gamma_\alpha] = 0 \quad , \quad \alpha = 1, 2$$

$$\{\Gamma_1, \Gamma_2\} = 0$$

Define $\quad c \equiv \frac{1}{2}\left(\Gamma_1 + i\Gamma_2\right)$

$$\{ c, c^+ \} = 1$$

$$c\, c^+ = \frac{1}{4} (\Gamma_1 + i \Gamma_2)(\Gamma_1 - i \Gamma_2)$$

$$= \frac{1}{4} \Big(\Gamma_1^2 - i \Gamma_1 \Gamma_2 + i \Gamma_2 \Gamma_1$$
$$+ \Gamma_2^2 \Big)$$

$$= \frac{1}{4} (2 - 2i \Gamma_1 \Gamma_2)$$

$$= \frac{1}{2} (1 - i \Gamma_1 \Gamma_2)$$

$$c^+ c = \frac{1}{4} (\Gamma_1 - i \Gamma_2)(\Gamma_1 + i \Gamma_2)$$

$$= \frac{1}{4} \Big(\Gamma_1^2 + i \Gamma_1 \Gamma_2 - i \Gamma_2 \Gamma_1$$
$$+ \Gamma_2^2 \Big)$$

$$= \frac{1}{4} (2 + 2i \Gamma_1 \Gamma_2)$$

$$= \frac{1}{2} \left(1 + i \Gamma_1 \Gamma_2 \right)$$

$$\Rightarrow \quad c c^+ + c^+ c = 1 \quad \checkmark$$

$$c^2 = (c^+)^2 = 0$$

$$c^2 = \frac{1}{4} \left(\Gamma_1 + i \Gamma_2 \right) \left(\Gamma_1 + i \Gamma_2 \right) =$$

$$= \frac{1}{4} \left(\Gamma_1^2 + i \Gamma_1 \Gamma_2 + i \Gamma_2 \Gamma_1 - \Gamma_2^2 \right) = 0$$

c : a (spinless) fermion operator.

peculiarity: delocalised in space.

$c^+ | GS \rangle$ is an eigenstate of $\mathcal{H}$, of zero excitation energy

$[\mathcal{H}, c^+] = 0$ because

$[\mathcal{H}, \Gamma_\alpha] = 0$.

$$\mathcal{H}\left(c^+ |GS\rangle\right) =$$

$$= c^+ \mathcal{H} |GS\rangle$$

$$= c^+ E_{GS} |GS\rangle$$

$$= E_{GS} \left(c^+ |GS\rangle\right)$$

A spinless fermion state can be occupied or empty.

The occupation of the delocalised fermion state allows to encode information:

empty $\rightarrow |0\rangle$

filled $\rightarrow |1\rangle$

Some advantages of this qubit:

(i) $|0\rangle$ and $|1\rangle$ are degenerate in energy $\Rightarrow$ no energy relaxation from $|1\rangle$ to $|0\rangle$.

(ii) Information is encoded nonlocally.

$\rightarrow$ robustness under local noise.

One problem: $|0\rangle$ and $|1\rangle$ have different fermion parity.

Fermion parity operator:

$$P_{12} \equiv 1 - 2c^{+}c = -i\,\Gamma_1 \Gamma_2$$

$$P_{12}|0\rangle = |0\rangle$$

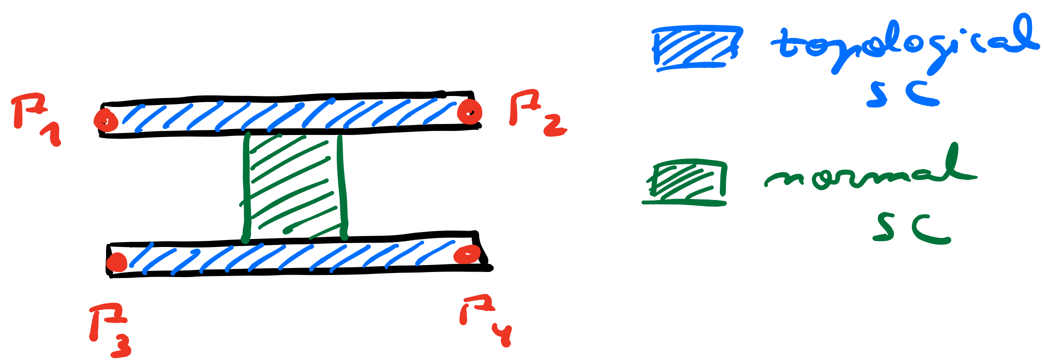
$c^+c|0\rangle = 0$

$$P_{12}|1\rangle = -|1\rangle$$

$c^+c|1\rangle = |1\rangle$

$\Rightarrow$ need single-electron transfers to manipulate the information, but a SC GS is made of pairs of e^--s.

A better qubit (actively pursued by Microsoft):



Two nonlocal fermions:

$$c_A = \frac{1}{2}\left(\Gamma_1 + i\Gamma_2\right)$$

$$c_B = \frac{1}{2}\left(\Gamma_3 + i\Gamma_4\right)$$

Occupations : $+$: occupied

$-$: empty

4 degenerate GS :

$$\left\{ \begin{array}{ll} |+,+\rangle & \multirow{2}{*}{} \\ |-,-\rangle & \\ |+,-\rangle & \\ |-,+\rangle & \end{array} \right.$$

$|+,+\rangle$, $|-,-\rangle$ } even total fermion parity

$|+,-\rangle$, $|-,+\rangle$ } odd total fermion parity.

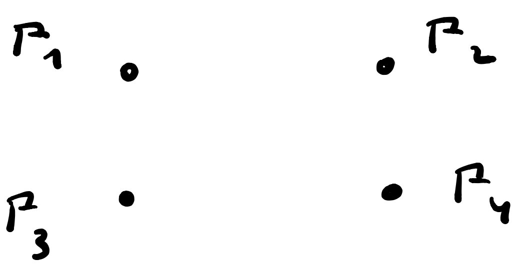
Work on a subspace of fixed total fermion parity. Then, there are two available states:

e.g.: $$|0\rangle = |+,+\rangle$$
$$|1\rangle = |-,-\rangle$$

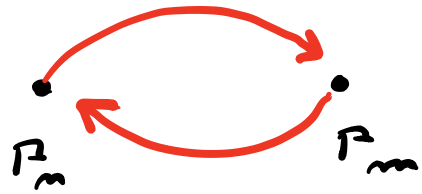
$|0\rangle$ and $|1\rangle$ differ by 2 electrons.

* Non-Abelian statistics:

(has not been verified yet in experiment)



Let U_{nm} be the unitary operator that exchanges Γ_n and Γ_m



Constraints on U_{nm} (heuristic):

(i) Must be slow (adiabatic), so that the system remains always in the degenerate GS.

(ii) Has to involve Γ_n and Γ_m. If the other MZM are "far", U_{nm} will only involve Γ_n and Γ_m.

(iii) Because $\Gamma_n^2 = \Gamma_m^2 = 1$, only the bilinear $\Gamma_n \Gamma_m$ will appear in U_{nm}.

(iv) Require:

$$U_{nm}^{\dagger} \, \Gamma_n \, U_{nm} \propto \Gamma_m$$

$$U_{nm}^{\dagger} \, \Gamma_m \, U_{nm} \propto \Gamma_n$$

(exchange)

Form of U_{nm} that satisfies all constraints:

$$\boxed{U_{nm} = \frac{1}{\sqrt{2}} \left(1 + \Gamma_n \Gamma_m \right)}$$

$$U_{nm}^{\dagger} \, \Gamma_n \, U_{nm} =$$

$$= \frac{1}{2} \left(1 + \Gamma_m \Gamma_n \right) \Gamma_n \left(1 + \Gamma_n \Gamma_m \right)$$

$$= \frac{1}{2}\left(P_n + P_m \underbrace{P_n^2}_{1} \right)\left(1 + P_n P_m \right)$$

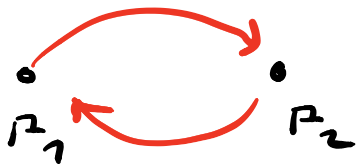
$$= \frac{1}{2}\Big(P_n + P_n^2 P_m + P_m$$
$$+ P_m P_n P_m \Big)$$

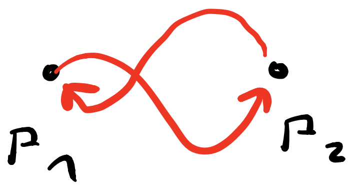
$$= \frac{1}{2}\Big(P_n + P_m + P_m$$
$$- \underbrace{P_m^2}_{1} P_n \Big)$$

$$= P_m$$

Likewise,

$$U_{nm}^{+} P_m U_{nm} = - P_n$$

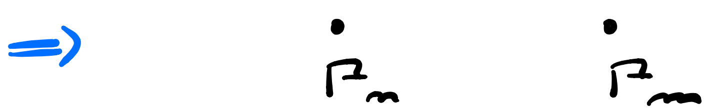




Independent of details of exchange.

This gives robustness to the operation.

Exchange twice:



$$U_{nm}^2 = \Gamma_n \Gamma_m$$

$$= i \, P_{nm}$$

$$P_{nm} = -i \, \Gamma_n \Gamma_m$$

$U_{nm}^2 \neq \mathbb{1}$! In undergraduate courses, we learn that exchanging the same pair of particles twice is like doing nothing ($U_{nm}^2 = \mathbb{1}$).

This then gives rise to fermionic and bosonic statistics. Clearly, MZM are neither fermions nor bosons; they are "anyons".

Moreover: $U_{12} U_{34} \neq U_{34} U_{12}$

$\Longrightarrow$ MZM are non-Abelian anyons.

* Logic operations (gates)

$$|0\rangle = |+,+\rangle$$

$$|1\rangle = |-,-\rangle$$

Nonlocal fermions:

$$c_A = \frac{1}{2}(\Gamma_1 + i\Gamma_2)$$

$$c_B = \frac{1}{2}(\Gamma_3 + i\Gamma_4)$$

Arbitrary state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$

Exchange MZM 1 and 2

$$U_{12} |\psi\rangle =$$

$$= \left[\alpha\, U_{12} |0\rangle + \beta\, U_{12} |1\rangle \right]$$

$$U_{12} |0\rangle = \frac{1}{\sqrt{2}} \left[1 + P_1 P_2 \right] |0\rangle$$

$$= \frac{1}{\sqrt{2}} \left[1 + i\, P_{12} \right] |0\rangle$$

$$= \frac{1}{\sqrt{2}} (1 - i) |0\rangle$$

$$U_{12} |1\rangle = \frac{1}{\sqrt{2}} (1 + i) |1\rangle$$

$$\Rightarrow U_{12} |\psi\rangle =$$

$$= \frac{1}{\sqrt{2}} \alpha (1 - i) |0\rangle$$

$$+ \frac{1}{\sqrt{2}} \beta (1 + i) |1\rangle$$

$$= \alpha \, e^{-i\frac{\pi}{4}} |0\rangle$$

$$+ \beta \, e^{i\frac{\pi}{4}} |1\rangle$$

If $|\psi\rangle \doteq \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, then,

$$U_{12} |\psi\rangle = \underbrace{\begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}}_{\substack{\text{single-qubit} \\ \text{rotation of } \frac{\pi}{4}.}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

If we instead exchange 2 and 3:

$$U_{23} |\psi\rangle = \alpha \, U_{23} |0\rangle$$

$$+ \beta \, U_{23} |1\rangle$$

$$U_{23}\,|0\rangle = \frac{1}{\sqrt{2}}\left(1 + \Gamma_2 \Gamma_3\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}}\Big[1 + \underbrace{i\left(c_A^{+} - c_A\right)}_{\Gamma_2}\left(c_B + c_B^{+}\right)\Big]\,|0\rangle$$

$$c_A = \frac{1}{2}\left(\Gamma_1 + i\Gamma_2\right)$$

$$c_A^{+} = \frac{1}{2}\left(\Gamma_1 - i\Gamma_2\right)$$

$$c_B = \frac{1}{2}\left(\Gamma_3 + i\Gamma_4\right)$$

$$c_B^{+} = \frac{1}{2}\left(\Gamma_3 - i\Gamma_4\right)$$

$$= \frac{1}{\sqrt{2}}\Big[1 + i\,c_A^{+} c_B + i\,c_A^{+} c_B^{+} - i\,c_A c_B - i\,c_A c_B^{+}\Big]\underbrace{|0\rangle}_{|+,+\rangle}$$

$$= \frac{1}{\sqrt{2}} \Big[|0\rangle + 0 + 0 + i\,|1\rangle + 0 \Big]$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle + i\,|1\rangle \right)$$

$$c_A\, c_B\, |+,+\rangle$$

$$= c_A\, c_B\, c_A^+\, c_B^+\, |-,-\rangle$$

$$= -\, c_A\, c_A^+\, c_B\, c_B^+\, |-,-\rangle$$

$$= \uparrow$$

$$c_B\, c_B^+ = 1 - c_B^+\, c_B$$

$$c_A\, c_A^+ = 1 - c_A^+\, c_A$$

$$= -\,(1 - c_A^+ c_A)(1 - c_B^+ c_B)\,|-,-\rangle$$

$$= -\left(1 - c_A^{+} c_A\right) |-,-\rangle$$

$$= -|-,-\rangle$$

Likewise,

$$U_{23} |1\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle + i |0\rangle \right)$$

Then,

$$U_{23} |\psi\rangle =$$

$$= \frac{1}{\sqrt{2}} \Big[\alpha \left(|0\rangle + i |1\rangle \right) + \beta \left(|1\rangle + i |0\rangle \right) \Big]$$

$$= \frac{1}{\sqrt{2}} \Big[(\alpha + i\beta) |0\rangle + (\beta + i\alpha) |1\rangle \Big]$$

$$U_{23}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} =$$

$$= \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}}_{\text{1 qubit gate}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$\equiv \sqrt{\text{NOT}}$

$$U_{23}^{2} \sim \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \text{"NOT"}$$

yet another operation:

$$U_{12}\, U_{23}\, U_{12} \doteq \frac{i}{\sqrt{2}}\underbrace{\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}_{\text{Hadamard gate}}$$

(allows to create superpositions
of states)

All of these logic operations are
topologically robust → no need
for error correction.

* One limitation:
Exchanging ("braiding")
MZM is not sufficient for
universal quantum computation.

Possible solution: use topological
phases whose excitations are
"Fibonacci anyons", which are
yet to be realised in the lab.
Proposals exist, which require

combining superconductivity
and the fractional quantum
Hall effect.