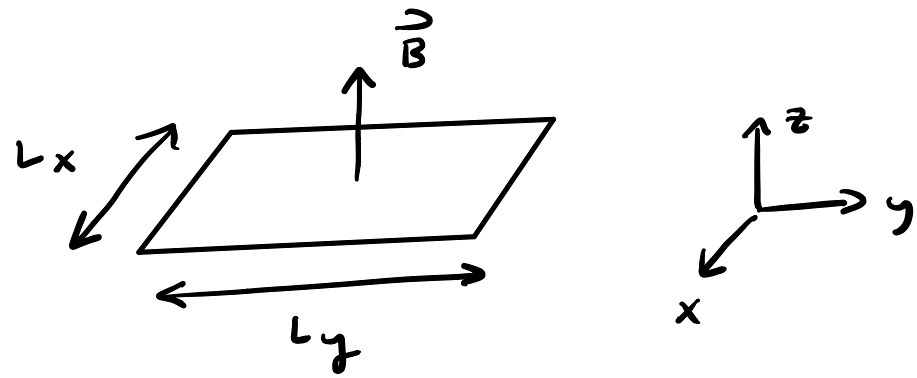
② Quantum Hall insulator

2.1 Free electrons in a magnetic field



Hamiltonian for one e^- :

$$\mathcal{H} = \frac{1}{2m} (\vec{p} - e\vec{A})^2$$

$$e = -|e|$$

$$\vec{A} = \hat{y} B x \quad \text{(Landau gauge)}$$

$$\Rightarrow \mathcal{H} = \frac{1}{2m} \left[p_x^2 + (p_y - eBx)^2 \right]$$

(1)

$$\mathcal{H}\,\Psi = E\,\Psi \qquad (2)$$

$$\Psi(x,y) = \frac{e^{iky}}{\sqrt{L_y}}\,\phi_k(x) \qquad (3)$$

$$k = \frac{2\pi}{L_y}\,\ell\,, \qquad \ell \in \mathbb{Z}$$

Combine (1), (2) and (3):

$$\underbrace{\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\,w_c^2\,(x - x_k)^2\right]}_{h(k)}\phi_k(x)$$

$$= E_k\,\phi_k(x)$$

where

$$w_c \equiv \frac{|e|\,B}{m} = \text{cyclotron frequency}$$

$$x_k \equiv \frac{\hbar k}{eB} = \text{guiding center}$$

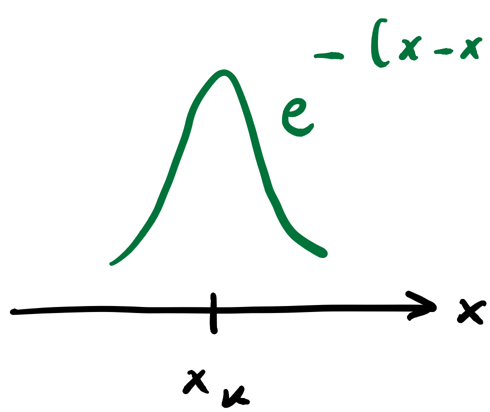
$$\left(\text{Note:} \quad x_k \in (0, L_x) \right)$$

$$\phi_{kn}(x) \sim e^{-(x-x_k)^2/2\ell_B^2} H_n(x - x_k)$$

where

$$\ell_B \equiv \sqrt{\frac{\hbar}{|e| B}} = \text{magnetic length}$$

$\sim$ localisation length



H_n : Hermite polynomials of degree n.

$$H_0(x - x_k) = 1$$

$$H_1(x - x_k) = x - x_k$$

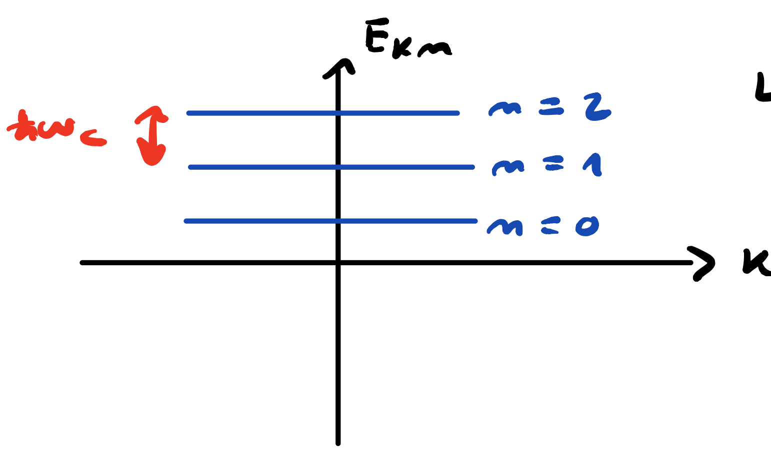
$$H_2(x - x_k) = (x - x_k)^2 - 1$$

...

(well-defined parity)

Eigenvalues: $E_{kn} = \hbar\omega_c\left(n + \frac{1}{2}\right)$

$n = 0, 1, 2, \ldots$



Landau levels

Degeneracy of each LL:

$$\frac{B L_x L_y}{\Phi_0},$$

where $\Phi_0 = \frac{h}{|e|}$ = quantum of flux

# of occupied LL:

density of e^--s / area

$$\nu = \frac{n_{2D} L_x L_y}{\frac{B L_x L_y}{\Phi_0}}$$

$$= \frac{n_{2D} \Phi_0}{B}$$

when ν integer, the system is an insulator, provided that

$k_B T \ll \hbar \omega_c$.

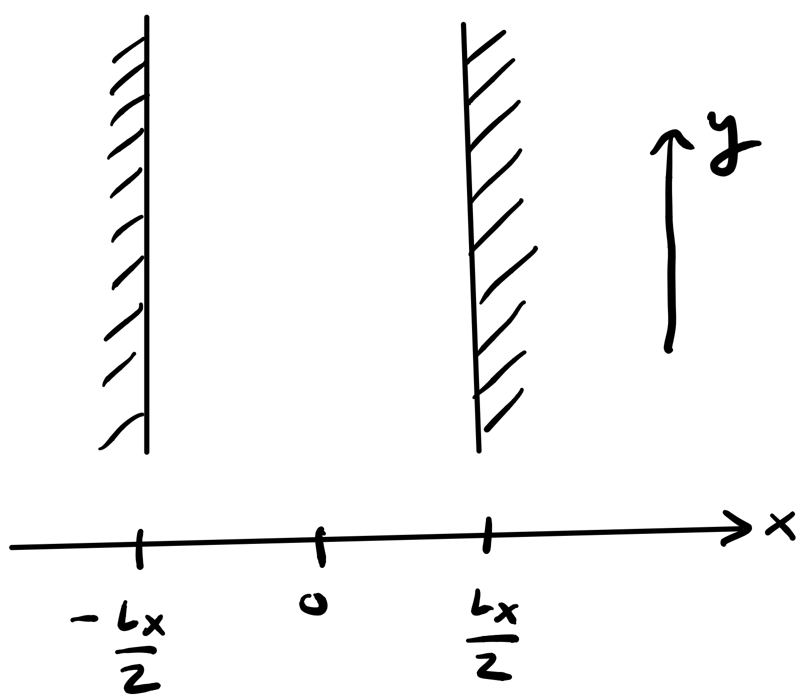
Classical regime: low B

$\Rightarrow \nu >>> 1$ (métal)

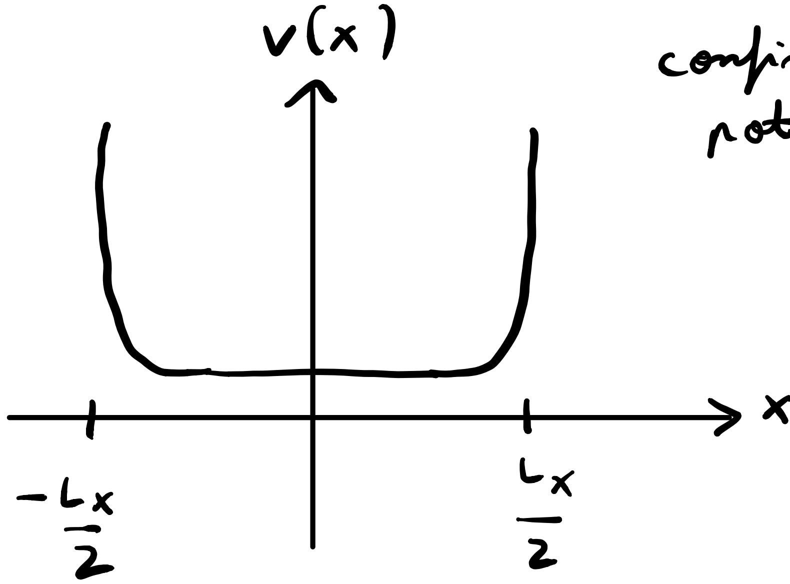
Quantum regime: high B

$\Rightarrow \nu \sim O(1)$ (insulator)

2.2 Chiral edge states



edges at $x = \pm L_x/2$



confinement potential

(preserves translational symmetry along y)

Recall: current density:

$$j_y = \frac{e}{A} \sum_{km} \underbrace{\langle \psi_{km} | v_y | \psi_{km} \rangle}_{\text{let's focus on this.}} f_{km}$$

$$\langle \psi_{km} | v_y | \psi_{km} \rangle =$$

$$= \int dx\, dy\, \psi^*_{km}(\vec{r})$$

$$\frac{1}{m}\left(-i\hbar \partial_y - eBx\right) \psi_{km}(\vec{r})$$

$$=$$

$$\psi_{km}(\vec{r}) = e^{iky} \phi_{km}(x) / \sqrt{L_y}$$

$$= \frac{1}{m} \underbrace{\frac{1}{L_y} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} dy}_{=1} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} dx$$

$$\left(\hbar k - eBx\right) \left|\phi_{km}(x)\right|^2$$

$$= \frac{-eB}{m} \int dx\, (x - x_k)\, |\phi_{km}(x)|^2$$

$$x_k = \frac{\hbar k}{eB}$$

$$= \omega_c \int dx\, (x - x_k)\, |\phi_{km}(x)|^2$$

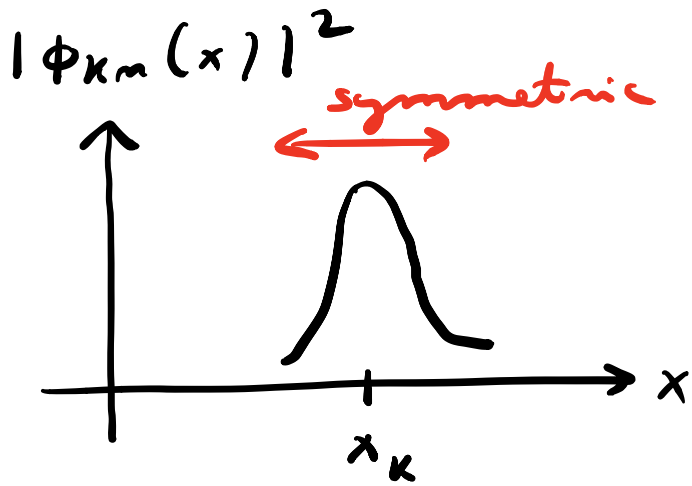
$$= \langle \psi_{km} | v_y | \psi_{km} \rangle$$

* Consider a value of k so that x_k is "far" (distance $\gg l_B$) from edges.

"bulk state"

Then, $V(x)$ can be neglected and $\phi_{km}(x) \sim e^{-(x-x_k)^2/2l_B^2}$

$$H_m(x - x_k)$$



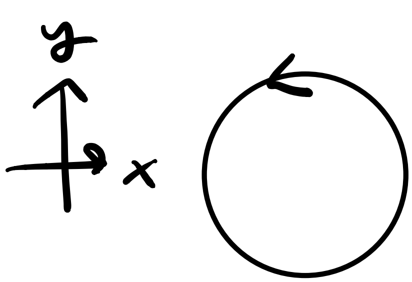
$$\Rightarrow \int dx\,(x - x_k)\,|\phi_{km}(x)|^2$$

$$\approx 0$$

$$\Rightarrow \langle \psi_{km} | v_y | \psi_{km} \rangle = 0$$

for bulk states.

(i) classical picture:



cyclotron orbit
No net velocity
along y.

(ii) Quantum picture:

If $V(x)$ negligible, then

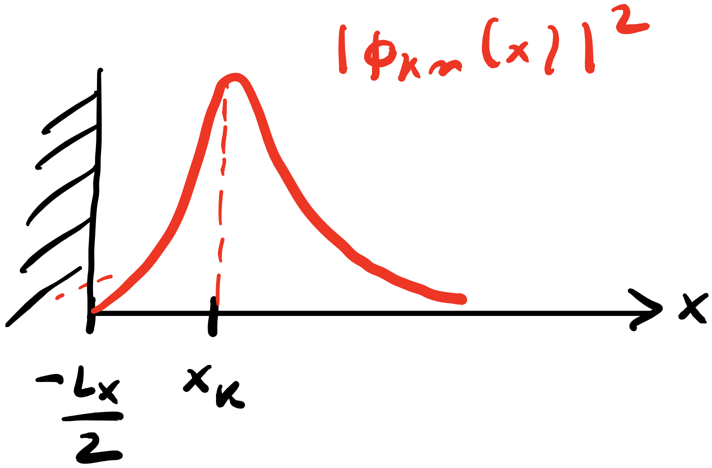
$$E_{kn} = \hbar\omega_c\left(n + \frac{1}{2}\right)$$

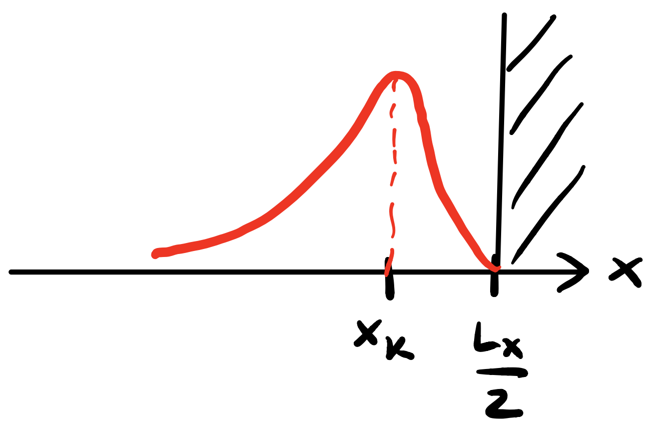
$$\Rightarrow \quad v_y \sim \frac{\partial E_{kn}}{\partial k} = 0$$

* Now, consider a value of k so that x_k is "close" (within a few l_B) to an edge.

"edge state"

Can no longer neglect $V(x)$.



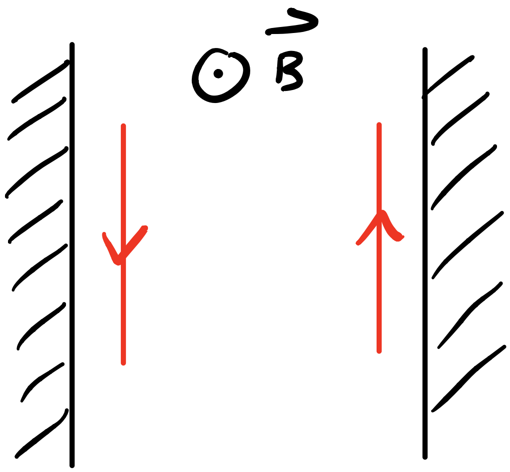


$|\phi_{km}(x)|^2$ has more weight away from the edge.

$$\Rightarrow \int dx\,(x - x_k)\,|\phi_{km}(x)|^2 \neq 0$$

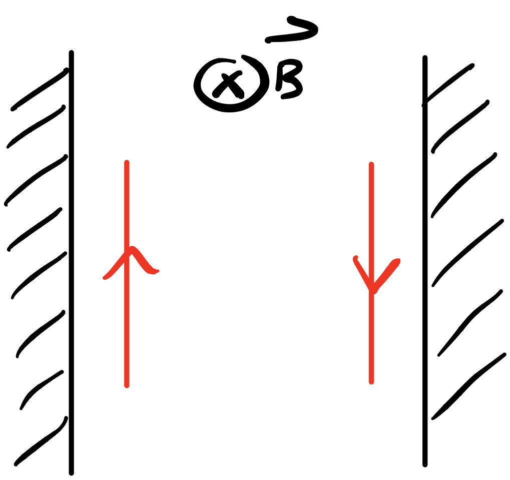
$\Rightarrow$

$$\langle \psi_{km} | v_y | \psi_{km} \rangle = \begin{cases} > 0, & x_k \approx -\frac{L_x}{2} \\ < 0, & x_k \approx \frac{L_x}{2} \end{cases}$$



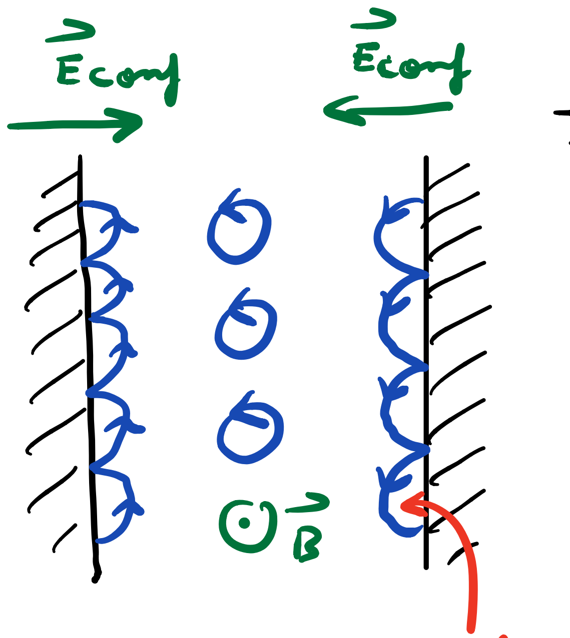
Propagating edge states

Similarly, one can show that



"chiral edge states"

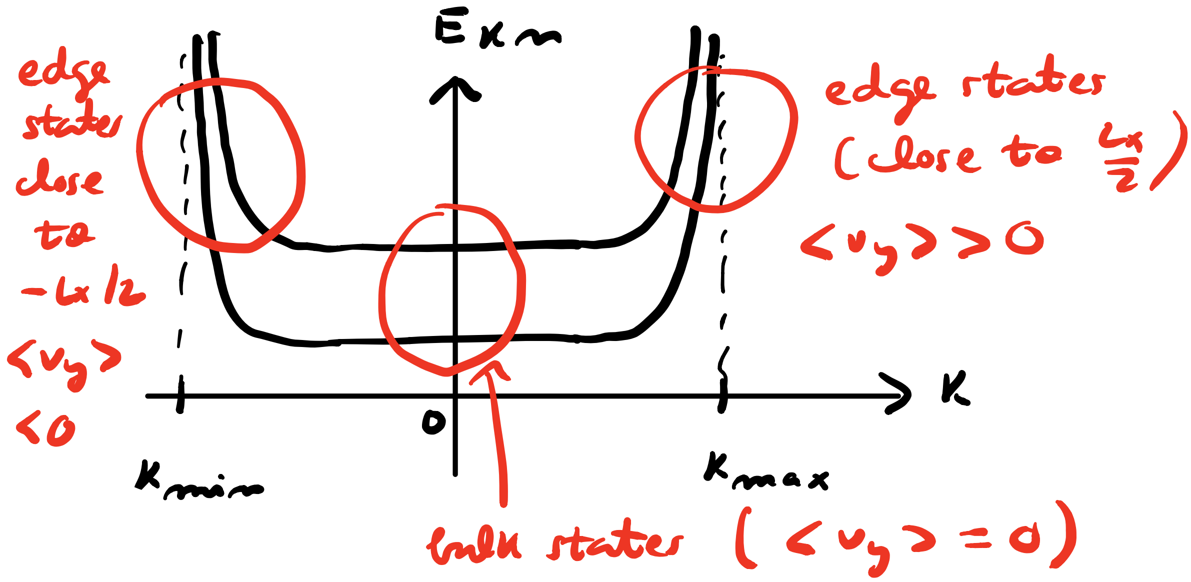
(i) classical picture:



$$\vec{E}_{conf} \equiv -\vec{\nabla} V$$

cycloid motion
$(\vec{E} \perp \vec{B})$

(ii) quantum picture:



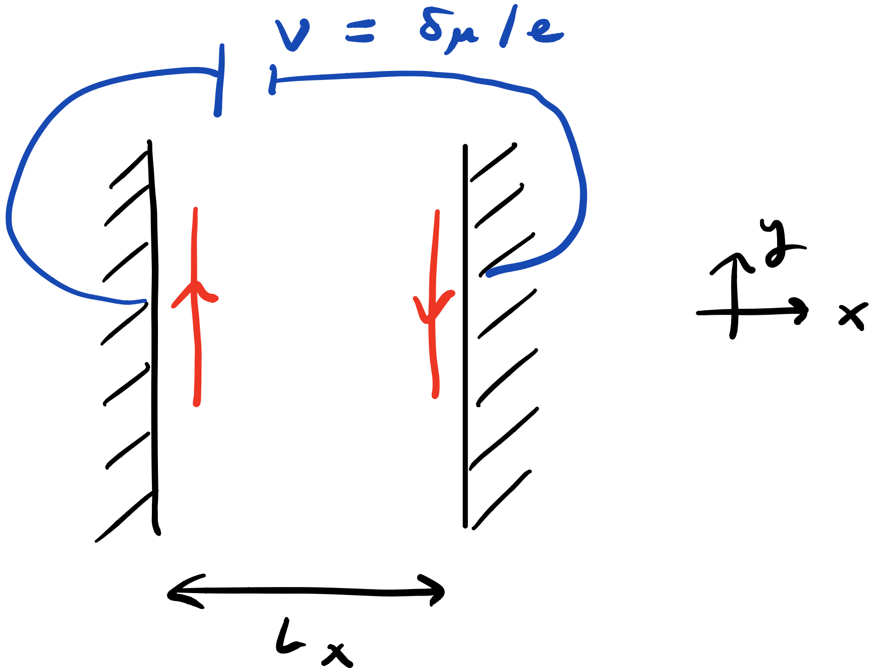
$$x_K = \frac{\hbar K}{eB} \in \left(-\frac{L_x}{2}, \frac{L_x}{2} \right)$$

$$K_{max} = \frac{L_x}{2} \frac{eB}{\hbar}$$

$$K_{min} = - K_{max}$$

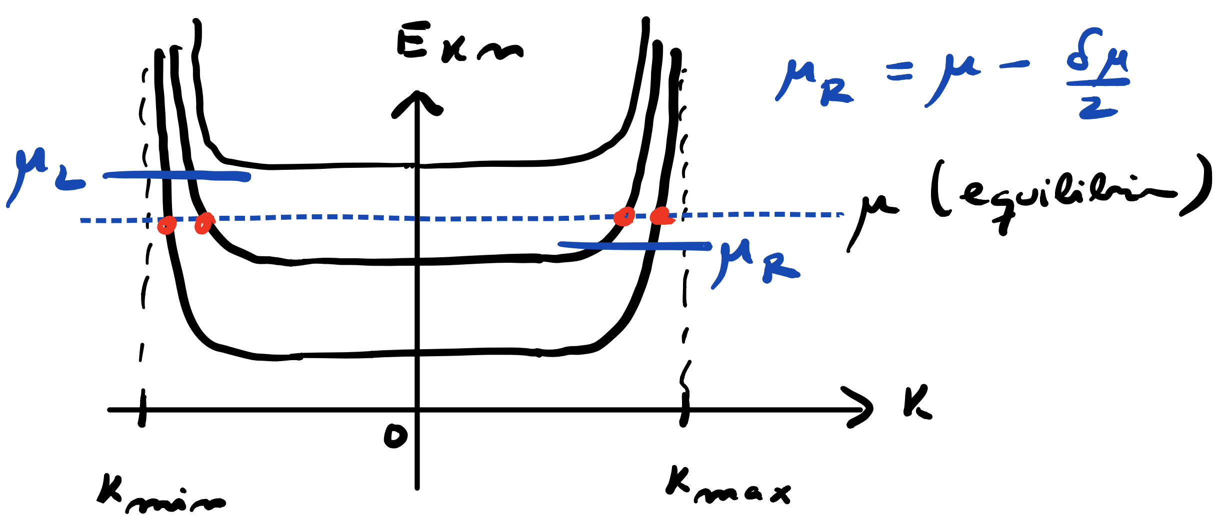
when $B \to -B$, edge states localized at $\approx \frac{L_x}{2}$ have $K < 0$ rather than $K > 0$.

2.3 Quantum Hall effect



$$\mu_L = \mu + \frac{\delta\mu}{2}$$

$$\mu_R = \mu - \frac{\delta\mu}{2}$$



Only edge states at Fermi level.

$$j_y = \frac{e}{A}\sum_{km}\langle\psi_{km}|v_y|\psi_{km}\rangle f_{km}$$

$$= \frac{e}{A}\sum_{\substack{k>0\\ m}}\langle\psi_{km}|v_y|\psi_{km}\rangle f_{km}$$

$$+ \frac{e}{A}\sum_{\substack{k<0\\ m}}\langle\psi_{km}|v_y|\psi_{km}\rangle f_{km}$$

$$= \frac{e}{A}\sum_{\substack{k>0\\ m}}\langle\psi_{km}|v_y|\psi_{km}\rangle \left[\underbrace{f^{(L)}_{km}}_{\text{occupation factor on one edge}} - \underbrace{f^{(R)}_{km}}_{\text{occupation factor on the other edge.}}\right]$$

$$f_{km}^{(L)} \rightarrow f(\mu_L) = f\left(\mu + \frac{\delta\mu}{2}\right)$$

$$\approx f(\mu) + \frac{\delta\mu}{2} \frac{\partial f}{\partial \mu}$$

$\uparrow$ $\delta\mu$ small

$$= f(\mu) - \frac{\delta\mu}{2} \frac{\partial f}{\partial E}$$

$$\Rightarrow f_{km}^{(L)} - f_{km}^{(R)} = -\delta\mu \frac{\partial f}{\partial E}$$

Then,

$$j_y = -\frac{e\,\delta\mu}{A} \sum_{\substack{k>0 \\ m}} \frac{\partial E_{km}}{\hbar\,\partial k} \frac{\partial f_{km}}{\partial E_{km}}$$

$$=$$

$\uparrow$

$$\sum_{k>0} \rightarrow \frac{L_y}{2\pi} \int_{k>0} dk$$

$$= - \frac{e\,\delta\mu}{L_x L_y} L_y \sum_n \int \frac{dk}{2\pi\hbar} \frac{\cancel{\partial E_{kn}}}{\partial k} \frac{\partial f}{\cancel{\partial E_{kn}}}$$

$$= - \frac{e\,\delta\mu}{h L_x} \sum_n \underbrace{\int_0^{k_{max}} dk \, \frac{\partial f_{kn}}{\partial k}}_{f_{k_{max},n} - f_{0,n}}$$

$$= \begin{cases} -1, & \text{if } n \text{ crosses } \mu \\ 0, & \text{otherwise} \end{cases}$$

$$= \frac{e\,\delta\mu}{h L_x} \underbrace{\nu}_{\text{"\# of edge states at Fermi level"}}$$

$$= \frac{e^2}{h} \nu \frac{V}{L_x} = \frac{e^2}{h} \nu E_x$$

$$= j_y$$

$$\Rightarrow \boxed{\sigma_{yx} = \frac{e^2}{h} \nu}$$

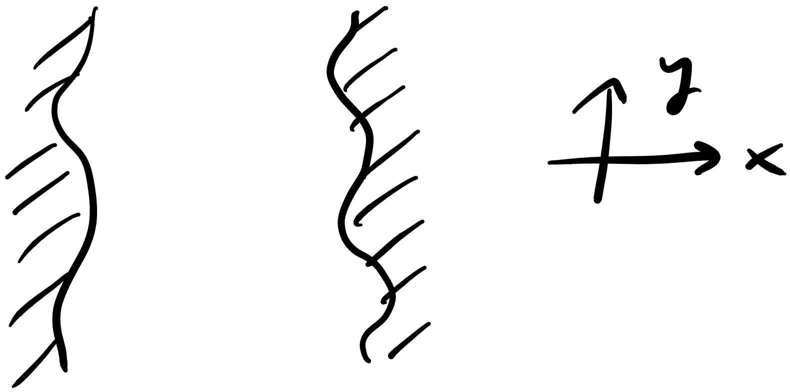
ν is also equal to the #
of occupied LL in the bulk.

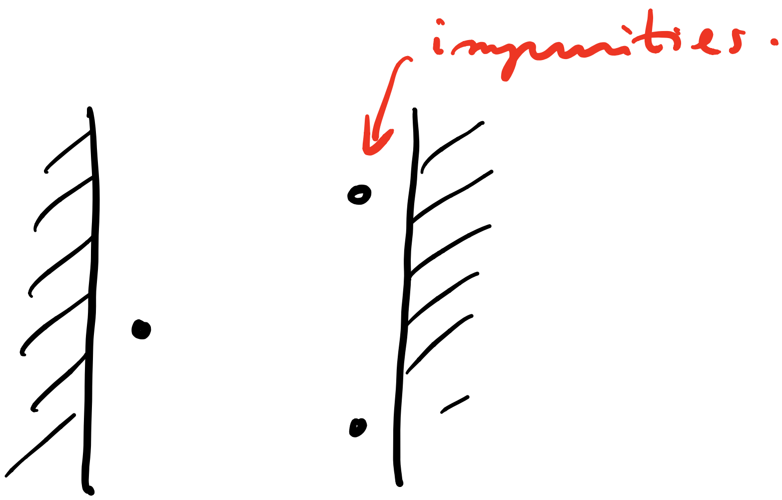
In this system,

$$\nu = \sum_{n \in occ} C_n .$$

So far, we have assumed
translational symmetry along
y. What if we relax this

assumption? Could σ_{yx} be degraded? $\sigma_{yx} \to \frac{e^2}{h} \nu T$ where $T < 1$?

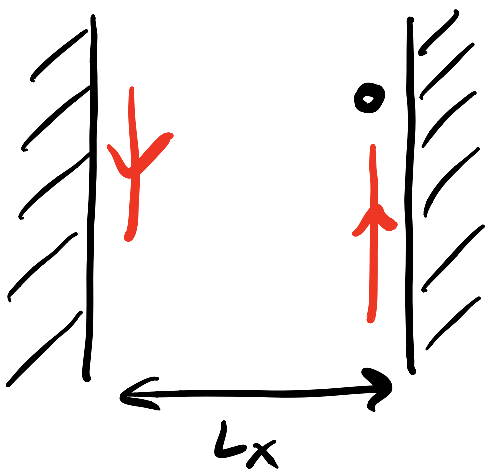




Answer: No, if the edges

are sufficiently distant from one another.

To degrade σ_{xy}, electrons need to be backscattered.



$$\langle \underbrace{\uparrow}_{\substack{\text{localised}\\ \text{on the}\\ \text{right}}} | \underbrace{V_{imp}}_{\substack{\text{local}\\ \text{in}\\ \text{space}}}(\vec{r}) | \underbrace{\downarrow}_{\substack{\text{localised}\\ \text{on the}\\ \text{left}}} \rangle$$

$$\simeq 0 \text{ if } L_x \gg \ell_B$$

Also, edge - bulk elastic scattering is not possible b/c there are no bulk states at the Fermi level.

Inelastic scattering is also suppresed at low temperature.

* Summary of QH insulator

(i) $\vec{B} \neq 0$

(ii) Insulating bulk

(iii) $\left.\begin{array}{l} \nu \text{ metallic edge states} \\ \text{Robust under perturbations} \end{array}\right\}$

(iv) $\nu = \sum_n C_n$ "bulk - surface correspondence"

"Topological insulator"