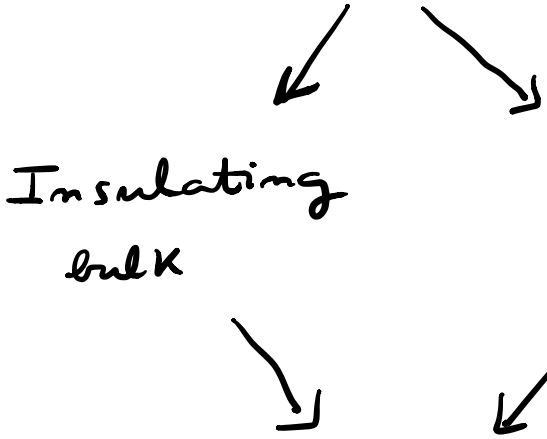
② Quantum Hall insulator

$\vec{B} \neq 0$



$$\sigma_{xy} = \nu \frac{e^2}{h}$$

$$\nu = \sum_{n \in \text{occ}} c_n$$

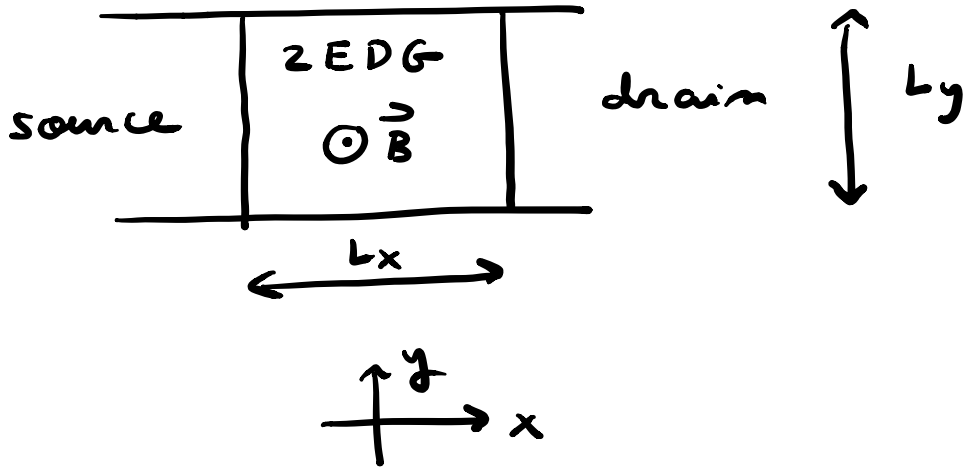
$$= \#\text{ of occupied LLs in bulk}$$

$$= \#\text{ of chiral edge states}$$

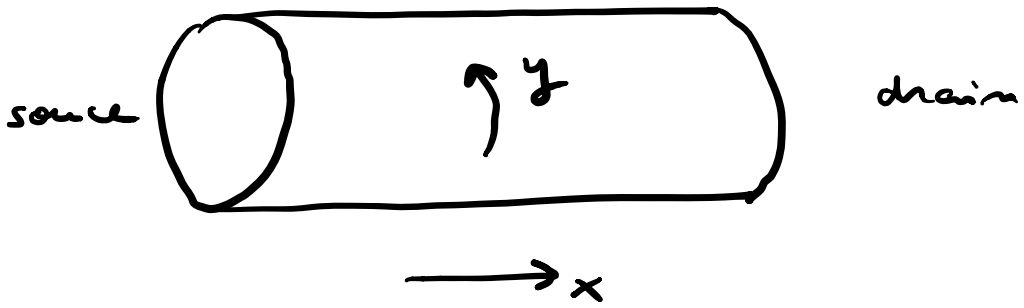
"Bulk-edge correspondence"

2.4 Laughlin's argument for Hall quantitation

[Laughlin, PRB 23, 5632 (1981)]



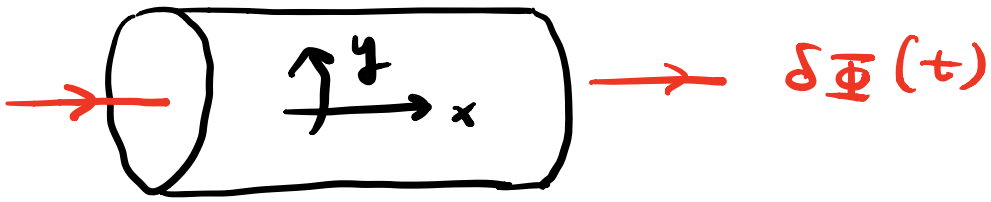
Assume periodic boundary conditions along y.



$\vec{B}$ is radial w.r.t. cylinder axis.

2DEG: surface of cylinder.

Insert a magnetic flux (adiabatically) through hole of cylinder.



Vector potential generated by flux:

$$\delta\vec{A}(t) = \frac{\delta\Phi(t)}{L_y}\,\hat{y}$$

Check:

$$\delta\Phi(t) = \oint_C \delta\vec{A} \cdot d\vec{\ell}$$

C ↑ loop around cylinder

$$= \oint dy \, \frac{\delta\Phi}{L_y} = \delta\Phi \quad \checkmark$$

The flux insertion produces an electric field:

$$\delta\vec{E}(t) = -\frac{\partial \, \delta\vec{A}}{\partial t}$$

$$= -\hat{y} \, \frac{1}{L_y} \, \frac{\partial}{\partial t} \, \delta\Phi \quad \| \, \hat{y}$$

Source - drain charge current density induced by $\delta \vec{E}$:

$$\delta j_x(t) = \sigma_{xy}\, \delta E(t)$$

$\delta j_x(t)$: A/m

σ_{xy}: intrinsic (independent of δE)

Charge transferred from source to drain, between $t = 0$ and $t = T$:

$$\delta Q = \oint dy \int_0^T dt\, \delta j_x$$

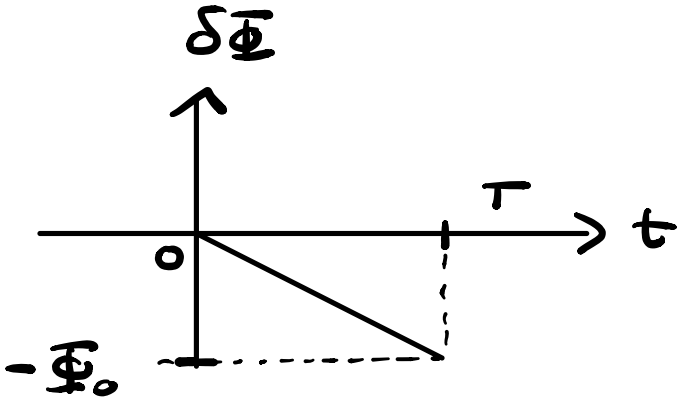
δQ: C

$$= -\sigma_{xy} \int_0^T \frac{\partial}{\partial t}\, \delta\Phi(t)$$

$$= -\sigma_{xy} \left[\delta\Phi(T) - \delta\Phi(0) \right]$$

σ_{xy} ↑ independent of $\delta\Phi$

Let's choose to insert a single quantum of flux



$$\Phi_0 = h/e$$

$$\Rightarrow \quad \delta Q = \sigma_{xy} \Phi_0$$

Gauge invariance : when inserting a flux quantum, one must have

$$\delta Q = n e \text{ , where}$$

$$n \in \mathbb{Z}.$$

$$\Rightarrow \quad n e = \sigma_{xy} \frac{h}{e}$$

$$\Rightarrow \quad \sigma_{xy} = n \frac{e^2}{h}$$

* "Proof" # 1 for $\delta\Phi = ne$

Total vector potential:

$$\vec{A} = \underbrace{Bx\,\hat{y}}_{\text{due to radial field}} + \underbrace{\delta\vec{A}}_{\text{due to flux insertion along cylinder}}$$

$$= \left(Bx + \frac{\delta\Phi}{L_y} \right) \hat{y}$$

$$\mathcal{H} = \frac{1}{2m} \left(\vec{p} - e\vec{A} \right)^2$$

$$\psi(x,y) = \frac{e^{iky}}{\sqrt{L_y}} \phi_k(x)$$

$$h(k)\ \phi_k(x) = E_k\ \phi_k(x)$$

where

$$h(k) = \frac{1}{2m}\left[p_x^2 + (\hbar k - eA)^2 \right]$$

$$= \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 (x - x_k)^2$$

where

x_k = guiding center

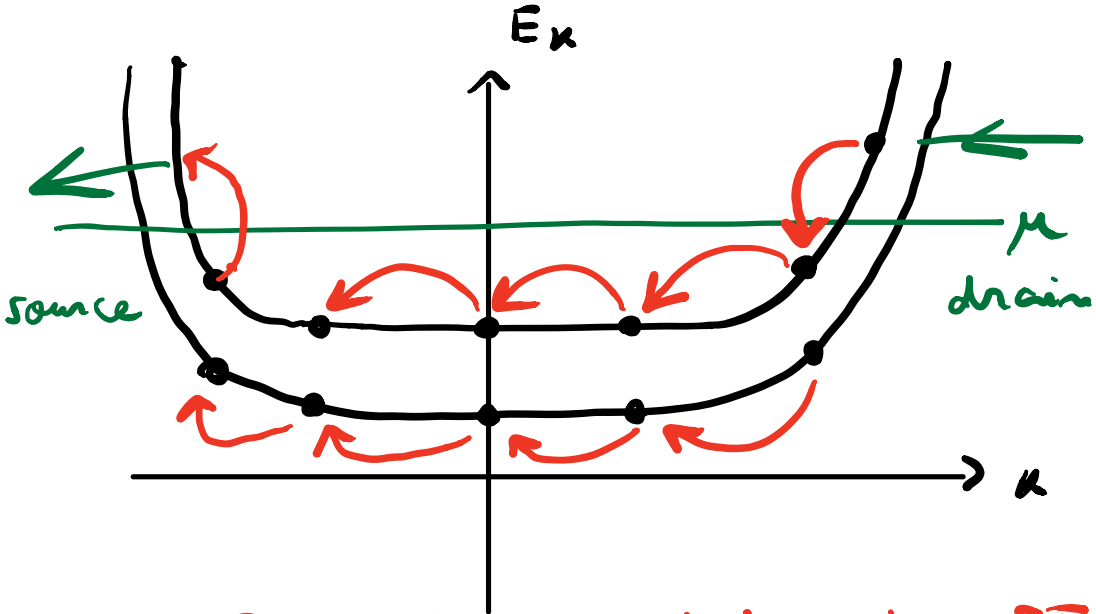
$$= \frac{\hbar k}{eB} + \frac{\delta\Phi}{L_y}$$

Now recall

$$k = \frac{2\pi}{L_y}\, l \,, \quad l \in \mathbb{Z}$$

$$\Rightarrow \underbrace{x_l}_{\text{center of wave function.}} = \frac{h}{eBL_y}\left(l - \frac{\delta\Phi}{\Phi_0}\right)$$

Having $\delta\Phi = \Phi_0$ is equivalent to $l \to l - 1$



↷ = flow of states when $\delta\Phi = \Phi_0$

For every value of x,

of e^-s = # of

occupied LLs = ν

Transferred # of e^-s upon

insertion of Φ_0 : ν

This proves $\delta Q = e\nu$

$\Rightarrow \sigma_{xy} = \nu \frac{e^2}{h}$

* Proof #2 for $\delta Q = ne$:

System must be periodic

in Φ, w/ periodicity Φ_0

(AB effect).

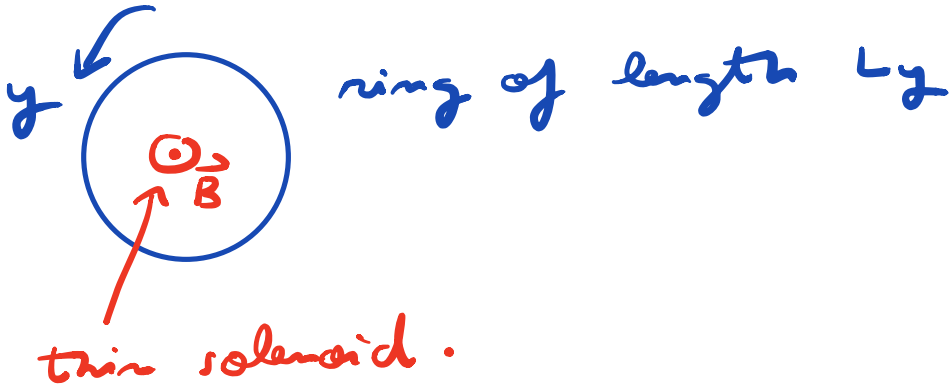
In other words, inserting Φ_0 gradually is like completing a full turn to Archimedes' screw.

Thouless: charge pumped in a cycle is equal to integer $\times e$.

$$\text{integer} = \sum_{n \in \text{Occ}} C_n .$$

In a QH insulator, this is precisely ν.

* Why the system is periodic in Φ with periodicity Φ_0 ?



Take $\vec{A} = A\,\hat{y}$

A independent of y.

$$\mathcal{H} = \frac{1}{2m}\left(p_y - eA\right)^2 + V(y)$$

$$V(y) = V(y + L_y)$$

$$\mathcal{H}\Psi = E\Psi$$

If $A = 0$, $\Psi(y) = \Psi(y + L_y)$

Let $\tilde{\Psi}(y) \equiv \Psi(y)\, e^{i \frac{2\pi}{\Phi_0} \int_0^y A\, dy'}$

$$= \Psi(y)\, e^{i \frac{2\pi}{\Phi_0} A y}$$

Replace this in Schrödinger equation:

$$\left[\frac{\hbar^2}{2m} + V(y) \right] \tilde{\Psi} = E \tilde{\Psi}$$

A disappeared? No.

It went to the boundary condition.

$$\tilde{\Psi}(y+L_y) = e^{i\frac{2\pi}{\Phi_0} A L_y}$$

$$\underbrace{e^{i\frac{2\pi}{\Phi_0} A y}\ \underbrace{\Psi(y+L_y)}_{\Psi(y)}}_{\tilde{\Psi}(y)}$$

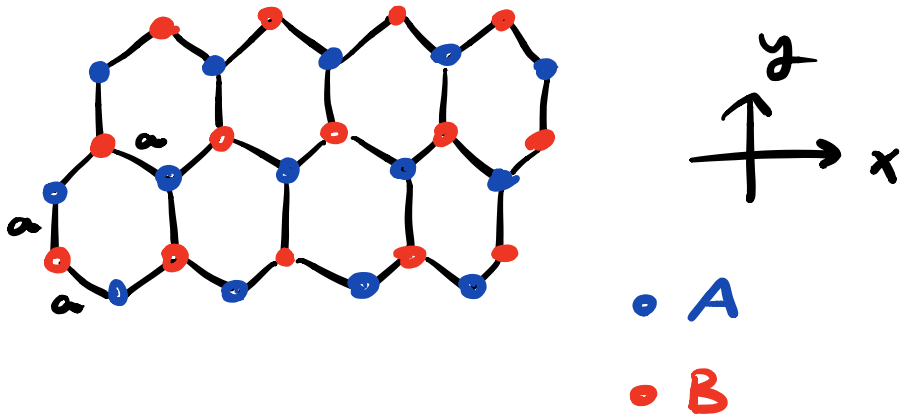
$$\tilde{\Psi}(y+L_y) = \tilde{\Psi}(y)\, e^{i\frac{2\pi}{\Phi_0} A L_y}$$

$$= \tilde{\Psi}(y)\, e^{i\frac{2\pi\,\Phi}{\Phi_0}}$$

Φ alters the solutions of the Schrödinger eq. unless $\Phi = n\Phi_0$. In the latter case, the flux completely disappears from the problem.

(3) Graphene

* Hexagonal lattice of C atoms



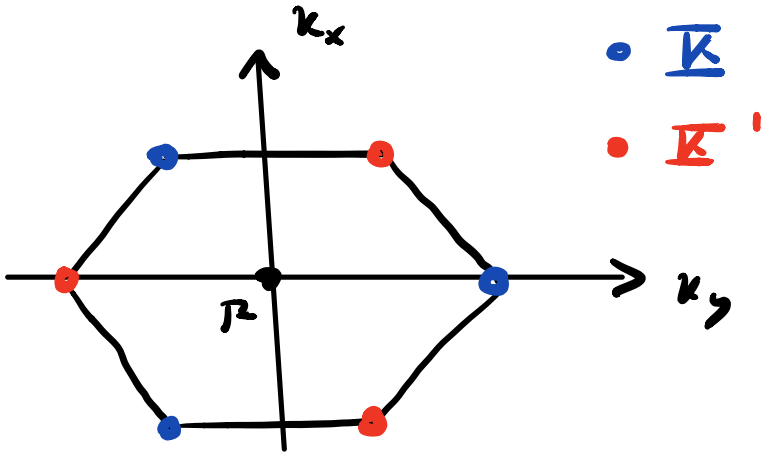
Two atoms per unit cell.

In graphene, A and B are carbon atoms.

BN : A = Boron

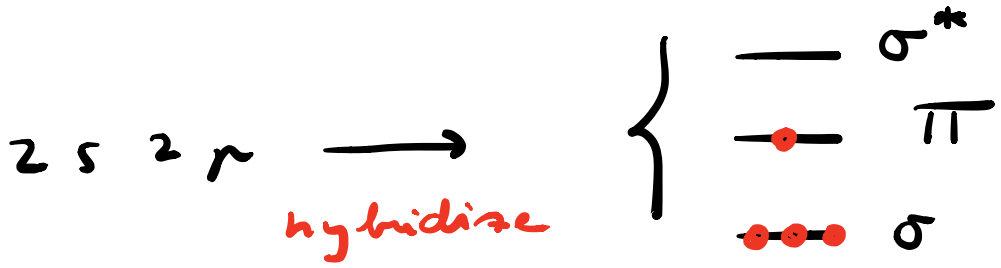
B = Nitrogen.

* 1 BZ



3.1 Low-energy electron states

* C : $1s^2\ 2s^2\ 2p^2$



π orbital is partically filled.

⇒ low-energy physics is associated to π orbital.

* Minimal tight-binding model (one orbital per site):

$$\mathcal{H}_0 = -t \sum_{\langle ij \rangle} \left(c_i^+ c_j + h.c. \right)$$

$\langle\ \rangle$: nearest neighbor.

For now, neglect the spin of e^-s.

Fourier transform:

$$H_0 = \sum_{\vec{k}} \left(c^{+}_{\vec{k}A},\ c^{+}_{\vec{k}B} \right) h_0(\vec{k}) \begin{pmatrix} c_{\vec{k}A} \\ c_{\vec{k}B} \end{pmatrix}$$

where

$$h_0(\vec{k}) = \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

$\vec{\sigma}$: sublattice pseudospin

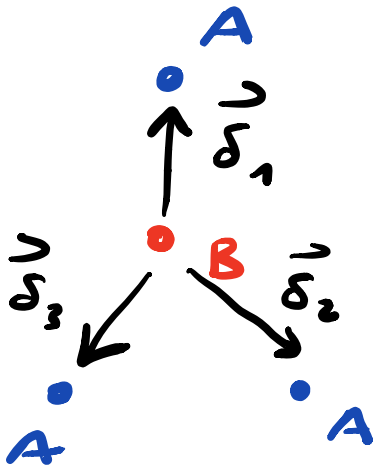
$\sigma_z \uparrow$: A

$\sigma_z \downarrow$: B

$$d_x(\vec{k}) = -t \sum_{j=1}^{3} \cos(\vec{k}\cdot\vec{\delta}_j)$$

$$d_y(\vec{k}) = -t \sum_{j=1}^{3} \sin(\vec{k}\cdot\vec{\delta}_j)$$

$$d_z(\vec{k}) = 0.$$



Nearest-neighbor vectors.

$$\vec{\delta}_1 = a(0, 1)$$

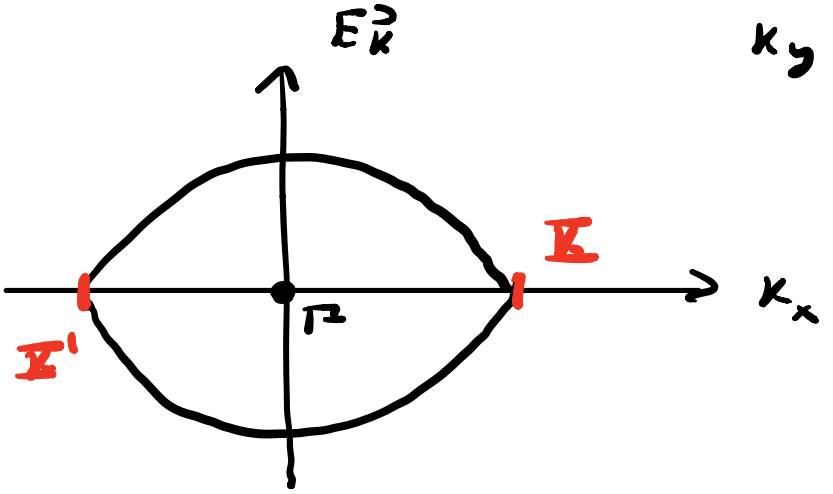
$$\vec{\delta}_2 = a\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

$$\vec{\delta}_3 = a\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

Energy eigenvalues:

$$E_{\vec{k}\pm} = \pm |\vec{d}(\vec{k})|$$

$$= \pm \sqrt{d_x(\vec{k})^2 + d_y(\vec{k})^2}$$



Bands touch at K and K'

⇒ at half-filling (one e^- per unit cell), graphene is a semimetal.

More precisely, it's a Dirac semimetal. Why Dirac?

Expand $\vec{k}$ near $\vec{K}$ and $\vec{K}'$:

(i) $\vec{k} = \vec{K} + \vec{q}$,

where $qa \ll 1$.

Then,

$$d_x \approx -\frac{3}{2} a t q_x \equiv v q_x$$

$$d_y \approx -\frac{3}{2} a t q_y \equiv v q_y$$

$$\Rightarrow \boxed{h_0^{(K)}(\vec{q}) = v(\sigma^x q_x + \sigma^y q_y)} \quad (1)$$

($\vec{q}$: momentum wrt to $\vec{K}$)

Massless 2D Dirac fermion,

with Dirac velocity v.

(ii) $\vec{K} = \vec{K}' + \vec{q}$

$$qa \ll 1 \qquad (2)$$

$$h_0^{(K')}(\vec{q}) \approx v\left(-\sigma^x q_x + \sigma^y q_y\right)$$

Massless 2D Dirac fermion.

Combine (1) and (2) into a 4×4 Hamiltonian:

$$h_{eff}(\vec{q}) = \begin{pmatrix} h_0^{(K)} & 0 \\ 0 & h_0^{(K')} \end{pmatrix}$$

$$= v\left(\tau^z \sigma^x q_x + \sigma^y q_y\right)$$

where $\vec{\tau}$ is the valley pseudospin.

$\tau^z \uparrow$: K

$\tau^z \downarrow$: K'

$h_{\text{eff}}(\vec{q})$ is the low-energy effective theory of graphene.

Dirac fermions may be gapped (they can acquire a nonzero mass) by breaking either space inversion symmetry or time-reversal symmetry.

Some gapped phases are topologically nontrivial.