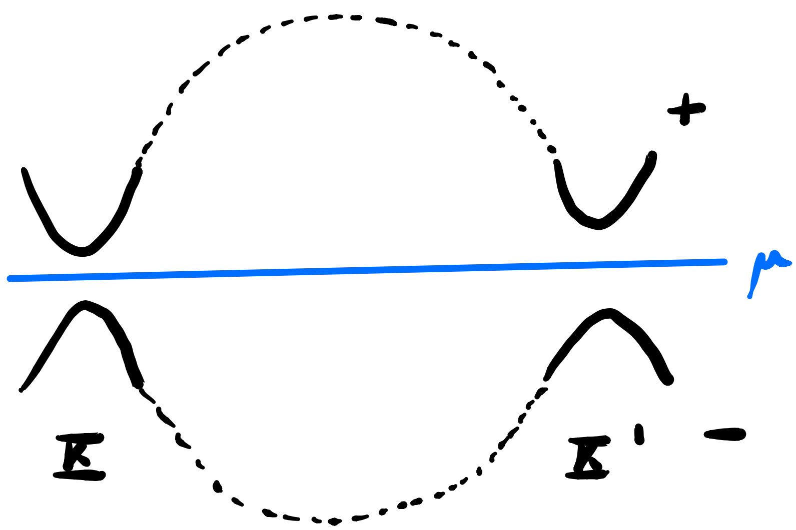
# 4 Quantum anomalous Hall insulator

a.k.a. $\left\{\begin{array}{l}\text{Chern insulator} \\ \text{Haldane } \quad "\end{array}\right.$

2D insulator w/ nonzero and quantized Hall conductivity, in the absence of a magnetic field.

## 4.1 Graphene (spinless)




$$h(\vec{k}) = \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

sublattice

* Chern # of occupied band:

$$C_- = \frac{1}{2\pi} \int_{1BZ} dk_x \, dk_y \; B_{xy,-}(\vec{k})$$

where (cf hw #3)

$$B_{xy,-}(\vec{k}) = \frac{1}{2} \hat{d} \cdot \left( \frac{\partial \hat{d}}{\partial k_x} \times \frac{\partial \hat{d}}{\partial k_y} \right)$$

* Calculation of $C_-$ from the low-energy theory:

$$h_{\mathbb{K}}(\vec{q}) = \vec{d}_{\mathbb{K}}(\vec{q}) \cdot \vec{\sigma}$$

where

$$d_{\mathbb{K},x}(\vec{q}) = v \, q_x$$

$$d_{K,y}(\vec{q}) = v\, q_y$$

$$d_{K,z}(\vec{q}) = \underbrace{m_K}_{\text{independent of } \vec{q}.}$$

$$h_{K'}(\vec{q}) = \vec{d}_{K'}(\vec{q}) \cdot \vec{\sigma}$$

$$d_{K',x}(\vec{q}) = -v\, q_x$$

$$d_{K',y}(\vec{q}) = v\, q_y$$

$$d_{K',z}(\vec{q}) = \underbrace{m_{K'}}_{\text{independent of } \vec{q}}$$

$$C_- = \underbrace{C_-^{(K)}}_{} + \underbrace{C_-^{(K')}}_{}$$

"partial Chern #s"

$$C_-^{(\lambda)} =$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dq_x \, dq_y \, B_{xy,-}^{(\lambda)} (\vec{q})$$

where $\lambda = K, K'$

$$C_-^{(K)} = \frac{1}{2} \operatorname{sgn}(m_K)$$

$$C_-^{(K')} = -\frac{1}{2} \operatorname{sgn}(m_{K'})$$

Then,

$$\boxed{C_- = \frac{1}{2}\left[\mathrm{sgn}(m_K) - \mathrm{sgn}(m_{K'})\right]}$$

* Recall:

$$m_K = \underbrace{u_z}_{\text{Semenoff mass}} + \underbrace{w_z}_{\text{Haldane mass}}$$

$$m_{K'} = u_z - w_z$$

If TRS is present, $w_z = 0$ and hence $\mathrm{sgn}(m_K) = \mathrm{sgn}(m_{K'})$

$\Rightarrow \boxed{C_- = 0}$ for Semenoff insulator.

This insulator is adiabatically connected to vacuum:

Keep $\mu_z \neq 0$ while taking all interatomic hoppings to zero. The final state is a vacuum. The energy gap remained always open in the process.

If TRS is broken (but SIS is preserved):

$$\mu_z = 0$$

$$w_z \neq 0$$

$$\Rightarrow \quad m_K = w_z = -m_{K'}$$

$$\Rightarrow \quad \mathrm{sgn}(m_K) = -\,\mathrm{sgn}(m_{K'})$$

$$\Rightarrow \quad \boxed{C_- = \mathrm{sgn}(m_K) = \pm 1}$$

Haldane insulator is topologically nontrivial.

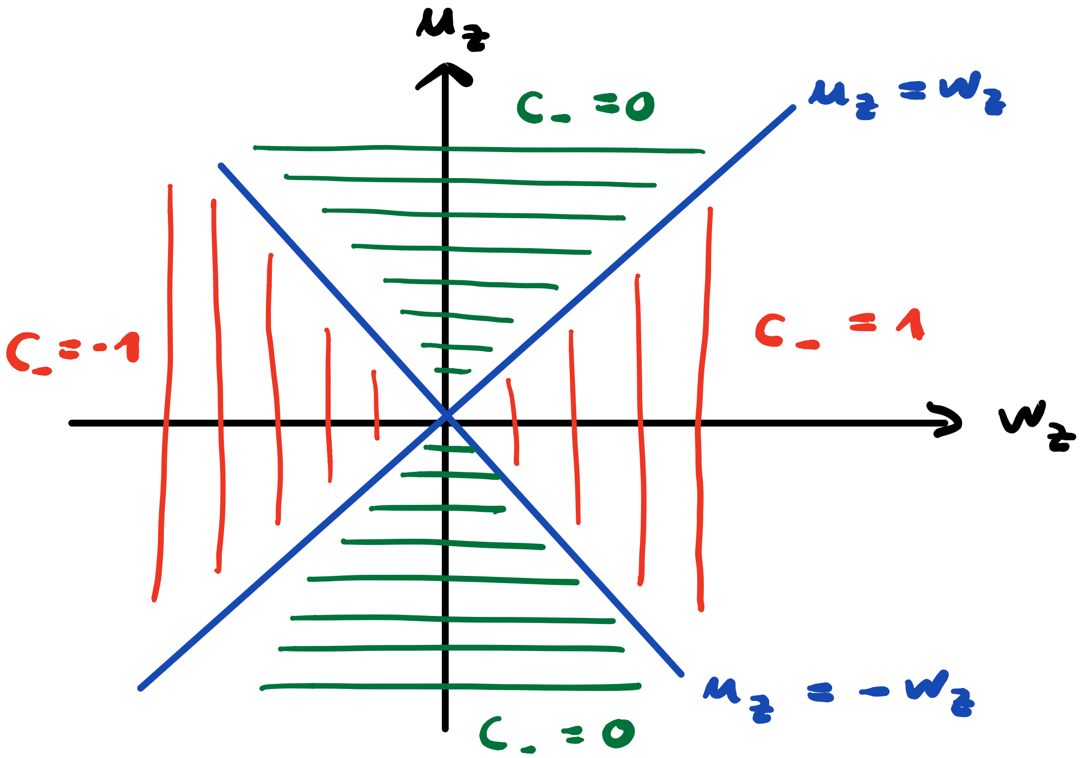
$$\sigma_{xy} = \frac{e^2}{h} C_- \neq 0 .$$

If both TRS and SIS are broken,

$$m_K = M_z + W_z$$

$$m_{K'} = M_z - W_z$$

Topological phase diagram:

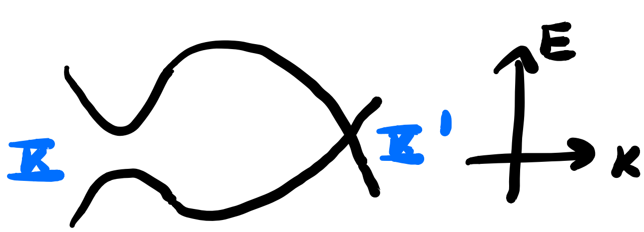



— $C_- = 0$

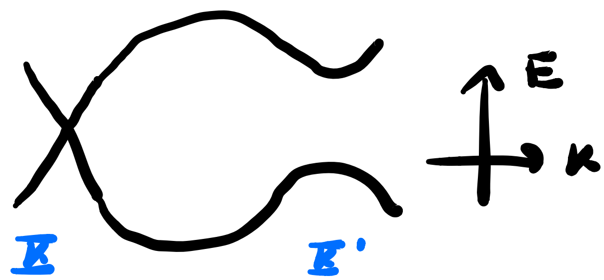
— $C_- \neq 0$

— gap closes (TPT)

$$u_z = w_z \Rightarrow m_K \neq 0 \,,\; m_{K'} = 0$$



$u_z = - w_z \Rightarrow m_{K'} \neq 0 , \; m_K = 0$




* Geometric interpretation of Chern # :

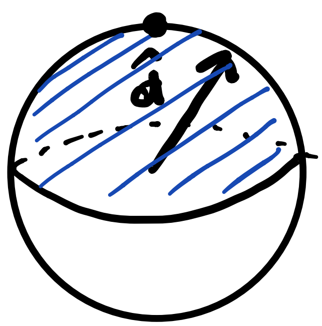
$h(\vec{k}) = \vec{d}(\vec{k}) \cdot \vec{\sigma}$ .

$C_-$ = # of times $\hat{d}(\vec{k})$ covers the unit sphere as $\vec{k}$ is varied within the 1st BZ. "winding #"

(i) semenoff insulator:

$d_z(\vec{k})$ doesn't change sign with $\vec{k}$.

$[\,\mathrm{sgn}(m_{I}) = \mathrm{sgn}(m_{I'})\,]$




case in which $d_z > 0$.

$\hat{d}$ only covers the northern hemisphere.

$\downarrow$

$C_- = 0$

$d_x = \pm v\, q_x$

$d_y = v\, q_y$

(ii) Haldane insulator :

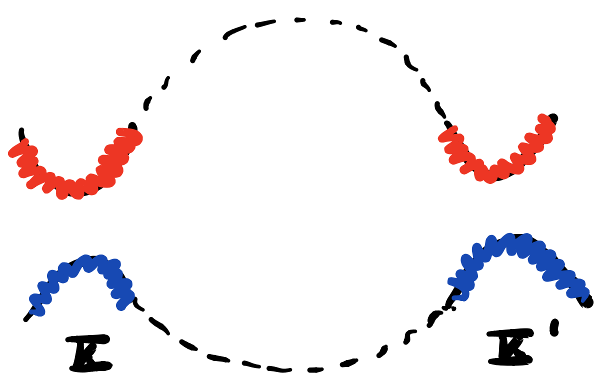
$d_z(\vec{x})$ has opposite sign near $K$ and $K'$ valleys.


$\hat{d}$ covers the whole sphere

$\Rightarrow |C_-| = 1$

* Concept of "band inversion" :

(i) Semenoff insulator



— states with predominantly A-like character

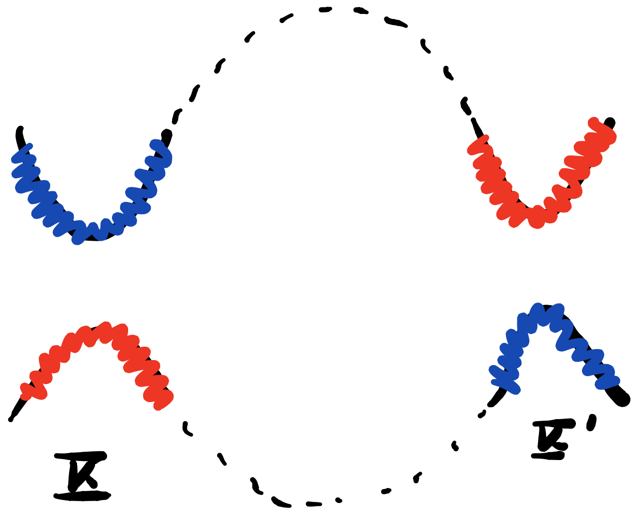
— states w/ predominantly B-like character.

$d_z \neq 0$

$\Big\downarrow$ TPT


Gap closing at K (say)

(ii) Haldane insulator




"Band inversion" at K

$\underbrace{\text{"Band inversion"}}$

↑

general mechanism for changing a topological invariant.

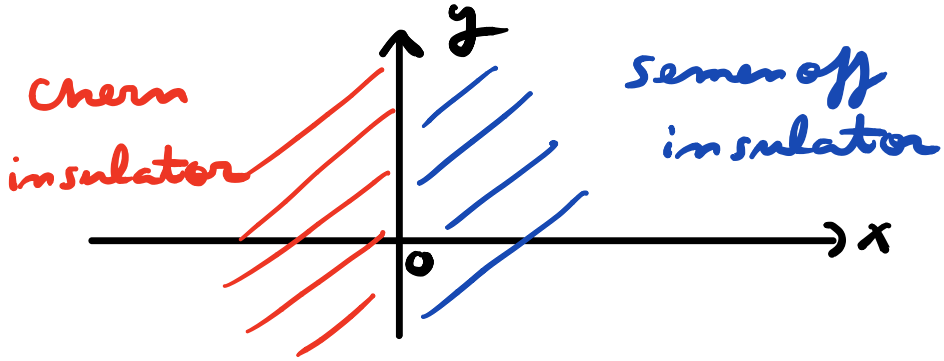
* Some caution words about low-energy calculation of $C_-$ :

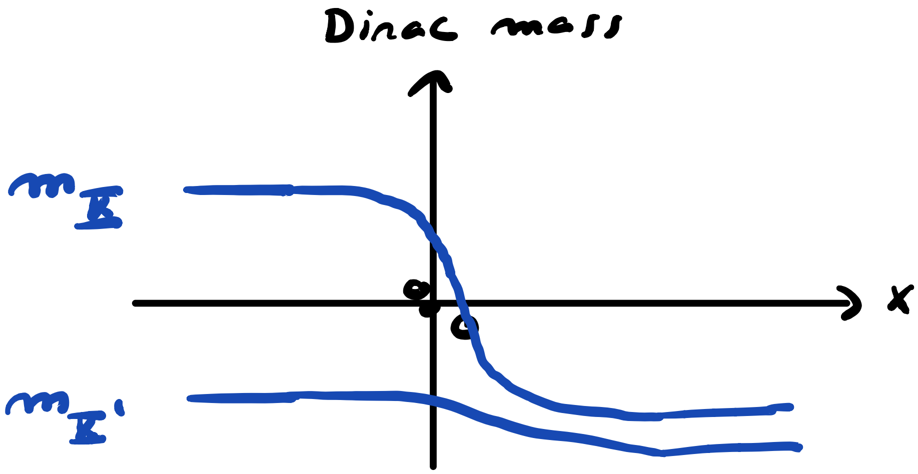
This method does not always work.

In general, one needs to consider the details of the high-energy electronic states in order to calculate $C_-$.

At any rate, the low-energy calculation of $C_-$ is good if we want to compute the change of Chern # across a band inversion.

## 4.2 Edge states






$m_K$ has a domain wall at $x = 0$.

Generalisation of Jackiw–Rebbi mode to 2D?

$$h_K = v\left(-i\sigma^x\partial_x - i\sigma^y\partial_y\right) + m_K(x)\,\sigma^z$$

$$[h_K, p_y] = 0$$

$$h_{\mathrm{I}}\, \psi(x,y) = E\, \psi(x,y) \quad (1)$$

where

$$\psi(x,y) = e^{iqy}\, \phi(x) \quad (2)$$

$q$ = momentum along $y$

= good quantum #

(1) and (2) :

$$v\left(-i\sigma^x \partial_x + \sigma^y q + m_{\mathrm{I}} \sigma^z\right) \phi(x)$$

$$= E_q\, \phi(x)$$

At $q = 0$, the problem becomes familiar:

$$\underbrace{v\left(-i\sigma^x\partial_x + m_{\underline{K}}\,\sigma^z\right)}_{\text{1D Dirac Hamiltonian w/ domain wall}}\phi(x) = E_0\,\phi(x)$$

We know that there's a zero-energy solution ($E_0 = 0$):

$$\phi_{edge}(x) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{\frac{1}{v}\int_0^x dx'\, m_{\underline{K}}(x')}$$

Exponentially localised at $x = 0$.

For $q \neq 0$ :

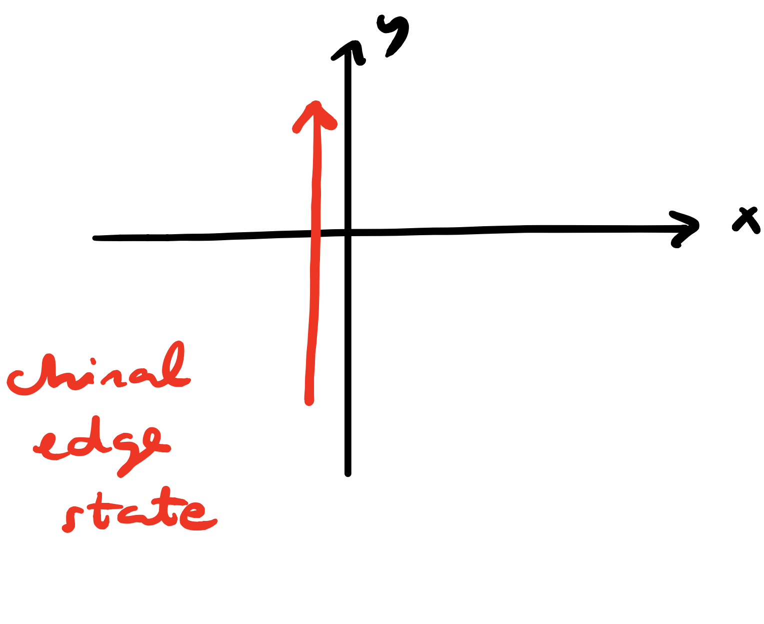
$$\psi_{edge}(x, y) = e^{iqy}\, \phi_{edge}(x)$$

Replace this in

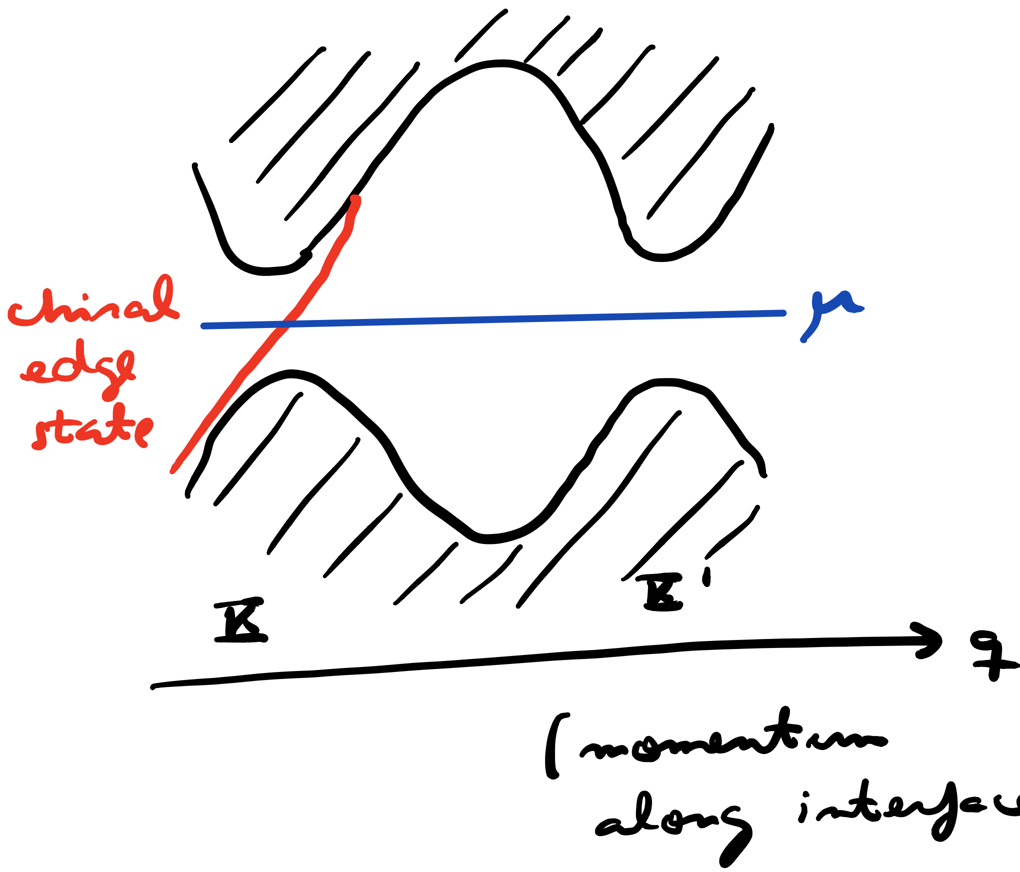
$$h_K\, \psi(x, y) = E\, \psi(x, y),$$

to get

$$E = E_q = v q$$



Band structure :




"1 D" unidirectional edge state.

Insulating bulk.

Conducting edge.

# of edge states = $|C_-|$

"bulk - boundary correspondence"

## 4.3 Experimental realisation

* Cold atoms in optical lattices:

  G. Jotzu et al., Nature 515, 237 (2014)

* Magnetically doped $Bi_2Se_3$ thin films.

  C.-Z. Chang et al., Science 340, 167 (2013)

* Essential ingredients:

  (i) 2D insulator

  (ii) Magnetism (to break TRS)

  (iii) Spin-orbit coupling

(communicates broken TRS from spin sector to orbital sector)

* Possible applications

(i) Chiral edge states

→ interconnects in integrated circuits.

(ii) other.

* Current limitation:

bulk gaps are small

→ QAH phase is realised only at low T.

In principle, there's no reason why this phase could not occur at room tempera-ture. More research is needed.