(4) Quantum anomalous Hall insulator

4.4 Calculation of Chern #s in arbitrary 2D insulators

* Thouless charge pump:

$$h(k,t) : \begin{cases} \text{periodic in } k \ \left(\text{period } \frac{2\pi}{a}\right) \\ \quad " \qquad " \quad t \ (\ " \quad T) \end{cases}$$

Assuming non-degenerate bands,

C_n = Chern # for band n

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \, A_n(k,T)$$

lec. 7

$$- \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dk \, A_n(k,0)$$

where $A_n(k,t)$ = Berry connection for band n.

* In a 2D insulator:

$$h(k_x, k_y) : \begin{cases} \text{periodic in } k_x \ \left(\frac{2\pi}{a}\right) \\ \text{'' \quad '' } k_y \ \left(\frac{2\pi}{a}\right) \end{cases}$$

(assume square lattice)

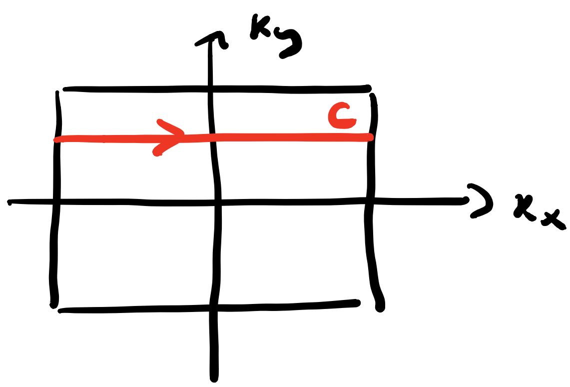
$$\boxed{C_n} = \frac{1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_x \, A_n\left(k_x, k_y = \frac{2\pi}{a}\right)$$

$$- \frac{1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_x \, A_n\left(k_x, k_y = 0\right)$$

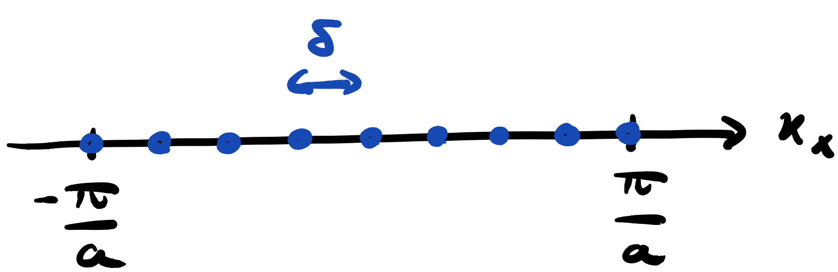
$$\equiv \boxed{\frac{1}{2\pi}\left[\gamma_n\left(\frac{2\pi}{a}\right) - \gamma_n(0)\right]}$$

where

$$\boxed{\underbrace{\gamma_n(k_y)}_{\text{Berry phase along } C} \equiv \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_x \, A_n(k_x, k_y)}$$



* Numerical calculation of $\gamma_n(k_y)$:



$$\gamma_n(k_y) = -\operatorname{Im}\ln\Big[$$

lec. 5

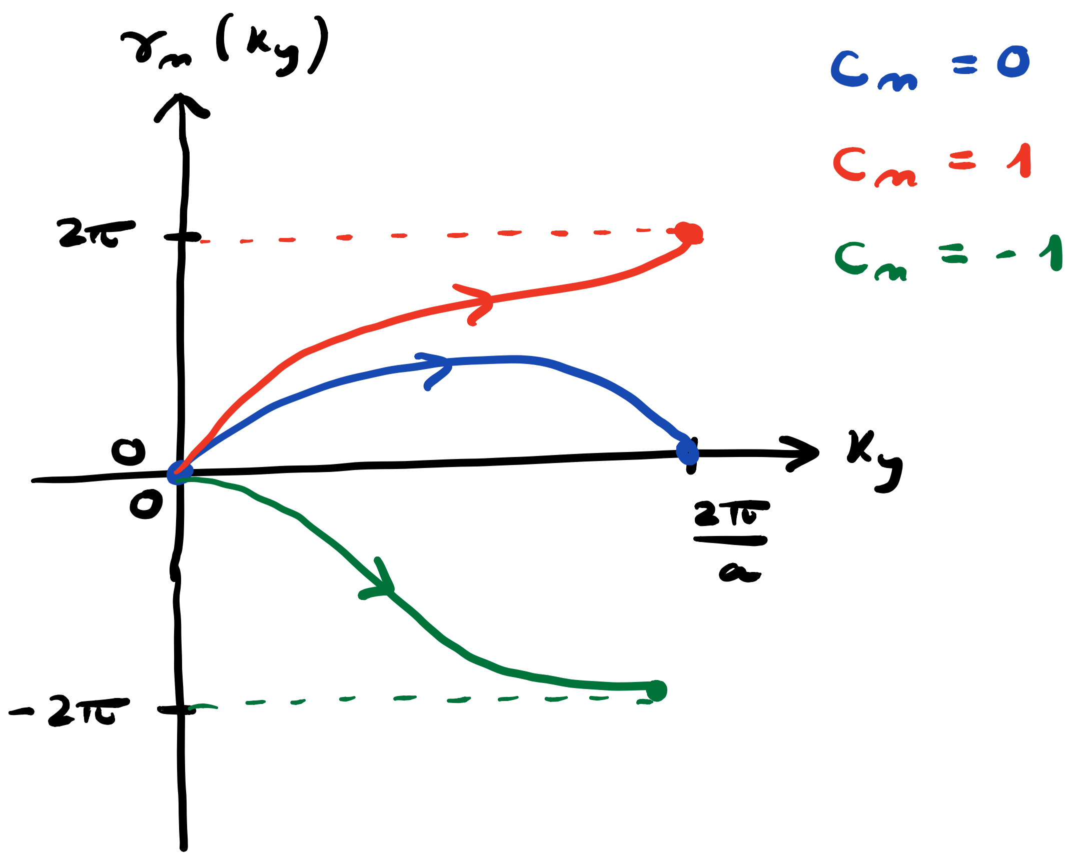
hw#2

$$\langle n, k_x = -\tfrac{\pi}{a}, k_y \,|\, n, k_x = -\tfrac{\pi}{a} + \delta, k_y \rangle$$

$$\langle n, k_x = -\tfrac{\pi}{a} + \delta, k_y \,|\, n, k_x = -\tfrac{\pi}{a} + 2\delta, k_y \rangle$$

$$\ldots \langle n, k_x = \tfrac{\pi}{a} - \delta, k_y \,|\, n, k_x = \tfrac{\pi}{a}, k_y \rangle \Big]$$

* Some possible outcomes:



* Link w/ Wannier centers:

(i) In the Thouless charge pump (lec. 7):

$$c_n = \frac{\langle 0, n | x | 0, n \rangle \big|_{t=T}}{a} - \frac{\langle 0, n | x | 0, n \rangle \big|_{t=0}}{a}$$

where

$$|0, n\rangle \propto \sum_{k \in 1BZ} |\psi_{kn}(t)\rangle$$

= Wannier state for band n, localised on unit cell 0.

x = position operator.

(ii) In a 2D insulator:

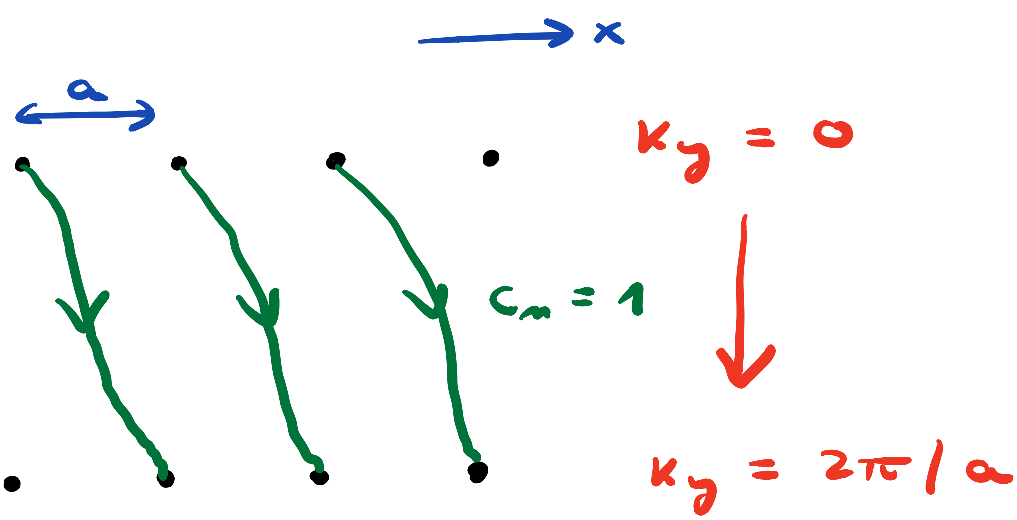
$$c_n = \frac{\langle 0,n | x | 0,n \rangle \big|_{k_y = 2\pi/a}}{a}$$

$$- \frac{\langle 0,n | x | 0,n \rangle \big|_{k_y = 0}}{a}$$

where

$$|0,n\rangle \propto \sum_{k_x \in 1BZ} |\psi_{k_x,n}(k_y)\rangle$$

For every k_y value, we have a 1D insulator :



$\rightarrow$: evolution of Wannier center

$\vec{E} \parallel \hat{y}$ produces charge transport along x

For degenerate bands, use non-Abelian Berry phase.

* Numerical package:

D. Gresch et al., PRB 95, 075146 (2017).

4.5 Valley Hall effect in Semenoff insulators

C_{K} = partial Chern # for valley K

$\neq 0$

$$C_{K'} = -C_K$$

$$\Rightarrow \quad C_K + C_{K'} = 0 \quad (\text{TRS})$$

$$\Rightarrow \quad \sigma_{xy} = \sigma_{xy,K} + \sigma_{xy,K'}$$

$$= 0$$

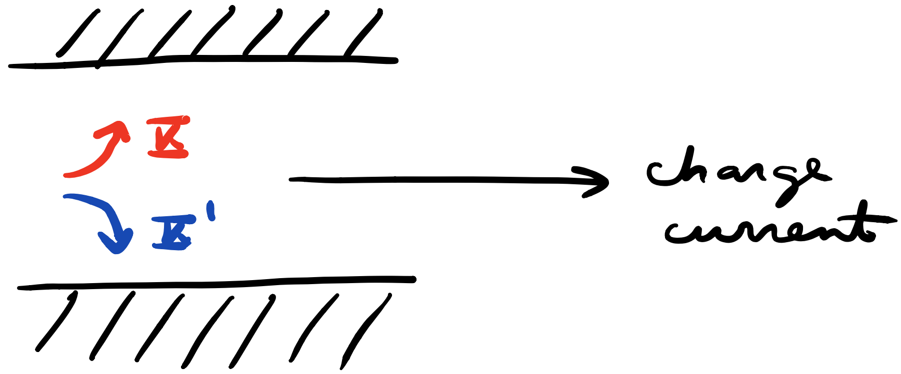
* Valley Hall conductivity:

$$\sigma_{xy}^{v} \equiv \sigma_{xy,K} - \sigma_{xy,K'}$$

$$= \frac{e^2}{h} C_K - \frac{e^2}{h} C_{K'}$$

$$= \frac{2e^2}{h} C_K \neq 0$$

Valley Hall effect (VHE):



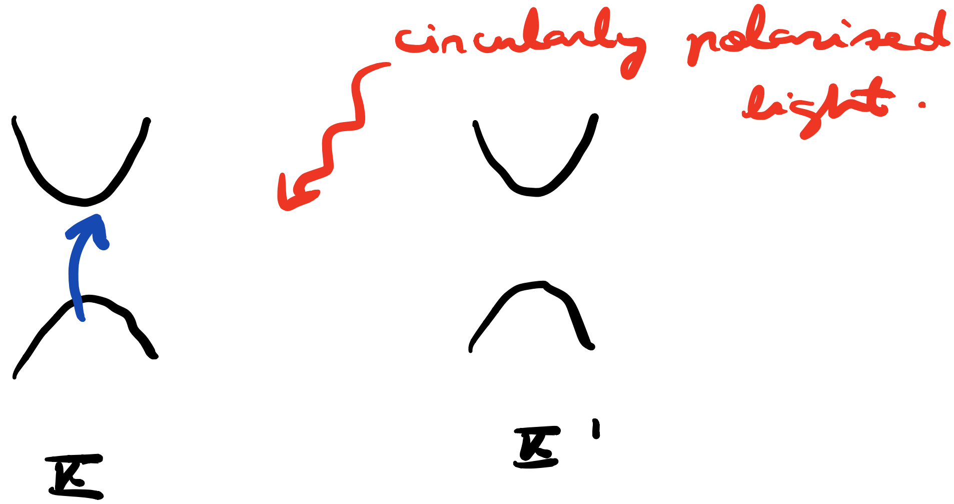
Effect is suppressed by disorder that induces intervalley scattering.

* Experimental detection of VHE:

K. F. Mak et al., Science 344, 1489 (2014)

Monolayer MoS_2 :

Semenoff insulator (breaks space inversion symmetry)



$\Rightarrow$ $\sigma_{xy,K}$ no longer cancels with $\sigma_{xy,K'}$

$\Rightarrow$ $\sigma_{xy} \neq 0$

Photoinduced Hall effect.

Bilayer MoS_2 : no effect.

Reason : bilayer MoS_2 has both TRS and SIS

$\Rightarrow$ Berry curvature $= 0$ for all $\vec{k}$

$$\Rightarrow C_{\underline{K}} = C_{\underline{K}'} = 0.$$

5 Quantum spin Hall insulator

Up until now, all topologically nontrivial phases in 2D broke TRS. In 2005, C. Kane and E. Mele realised that 2D insulators with TRS could also be topologically nontrivial.

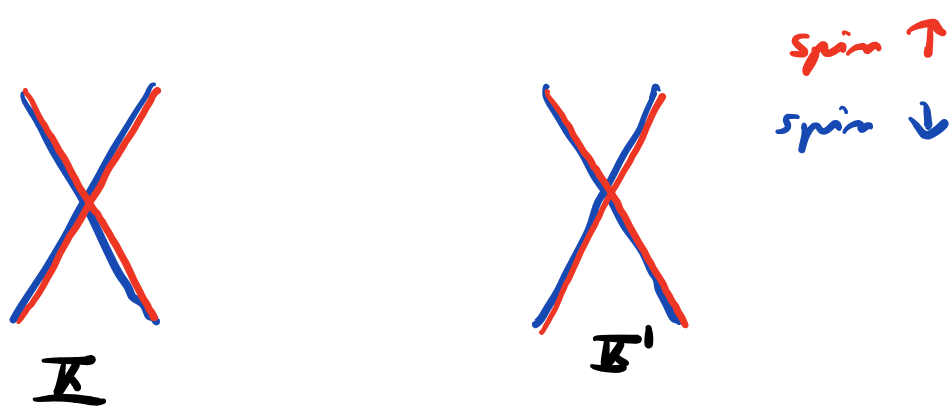
5.1 Kane-Mele model

* Spinful graphene without spin-orbit coupling:

$$\underbrace{h^{(0)}}_{\substack{\text{low-energy}\\ \text{effective}\\ \text{theory}}}(\vec{q}) = v\,(\underbrace{\tau^z}_{\text{valley}}\,\underbrace{\sigma^x}_{\text{sublattice}}\, q_x + \sigma^y q_y)\,\underbrace{s^0}_{\substack{\text{identity}\\ \text{in}\\ \text{spin}\\ \text{space}}}$$

$$= \begin{pmatrix} \overbrace{\boxed{h^{(0)}_{\uparrow}(\vec{q})}}^{4\times 4} & 0 \\ 0 & \underbrace{\boxed{h^{(0)}_{\downarrow}(\vec{q})}}_{4\times 4} \end{pmatrix}$$

Spin-degenerate Dirac cones.



* Gap-opening w/ spin-orbit coupling:

[C. Kane and E. Mele,
PRL 95, 226801 (2005)]

$$h(\vec{q}) = h^{(0)}(\vec{q}) + \delta h(\vec{q}),$$

$$\boxed{\delta h(\vec{q}) = \underbrace{\lambda_{SO}}_{\text{const}} \, s^z \sigma^z \tau^z}$$

This perturbation preserves all symmetries of graphene.

In particular:

$$\text{(i)} \quad \delta h(\vec{q}) = \Theta \, \delta h(-\vec{q}) \, \Theta^{-1}$$

with $\underbrace{\Theta = -i\, s^y K}_{\substack{\uparrow \\ \text{lecture 16}}}\, \tau^x$

s^y = y Pauli matrix

(ii) $\delta h(\vec{q}) = \Pi\, \delta h(-\vec{q})\, \Pi^{-1}$

with $\Pi = \tau^x \sigma^x$

Because it is allowed by symmetry, δh will always be present in graphene.

* Origin of δh in the tight-binding model of graphene:

$$\delta h = i \lambda_{SO} \sum_{\langle\langle i,j \rangle\rangle} c_i^{+} s^z c_j + h.c.$$

2nd neighbor

$$c_i = (c_{i\uparrow} , c_{i\downarrow})$$

$$c_i^{+} s^z c_j = c_{i\uparrow}^{+} c_{j\uparrow} - c_{i\downarrow}^{+} c_{j\downarrow}$$

For spinless electrons (w/o s^z), this reduces to the term added by Haldane in 1988.

spin-dependent hopping originates from spin-orbit coupling.

* Then,

$$h(\vec{q}) = \begin{pmatrix} h_\uparrow(\vec{q}) & 0 \\ 0 & h_\downarrow(\vec{q}) \end{pmatrix}$$

(still diagonal in spin)

$$h_\uparrow(\vec{q}) = h_\uparrow^{(0)}(\vec{q}) + \delta h_\uparrow(\vec{q})$$

$$= v\left(\tau^z \sigma^x q_x + \sigma^y q_y\right) + \lambda_{SO} \sigma^z \tau^z$$

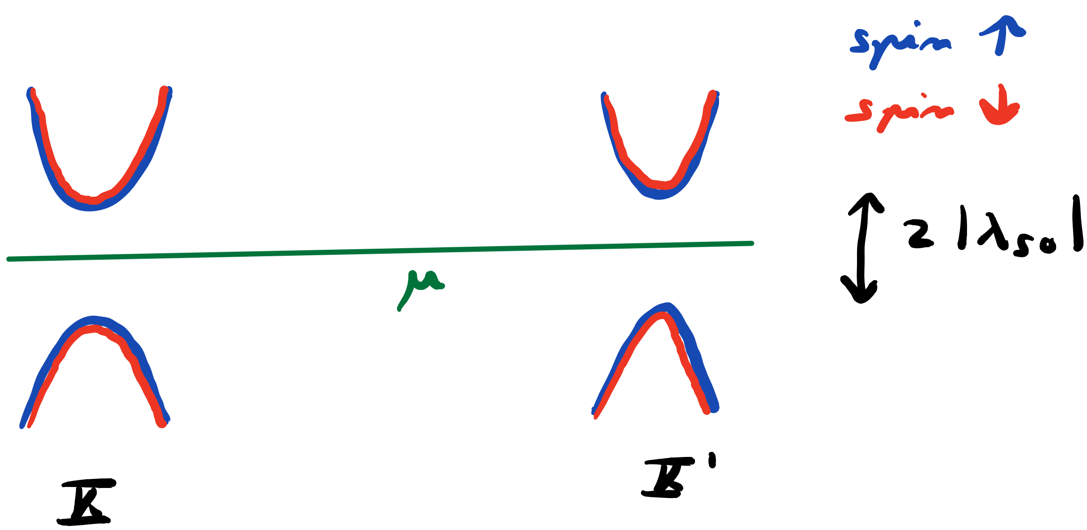
= Chern insulator w/ Haldane mass λ_{SO}.

$h_\downarrow(\vec{q}) =$

$$= v\left(\tau^z \sigma^x q_x + \sigma^y q_y\right) - \lambda_{SO} \sigma^z \tau^z$$

= Chern insulator w/ Haldane mass $-\lambda_{so}$.

So, we have two independent copies of a Chern insulator, with opposite Chern #s.



In sum, spin-orbit coupling opens a gap w/o breaking any symmetries.

* Chern # of occupied bands :

$$C = C_\uparrow + C_\downarrow$$

$$= C_{\uparrow, K} + C_{\uparrow, K'}$$

$$+ C_{\downarrow, K} + C_{\downarrow, K'}$$

$$= \mathrm{sgn}(\lambda_{SO}) + \mathrm{sgn}(-\lambda_{SO})$$

lecture 18

$$= 0$$

Not surprising b/c TRS is present.

Hence, $\sigma_{xy} = 0$.

But, we can define a

"spin Chern number" :

$$C_s \equiv C_\uparrow - C_\downarrow = 2\,\mathrm{sgn}(\lambda_{so})$$
$$\neq 0,$$

because each spin sector is topologically nontrivial.

What's the physical significance of C_s?

Define the spin-Hall conductivity:

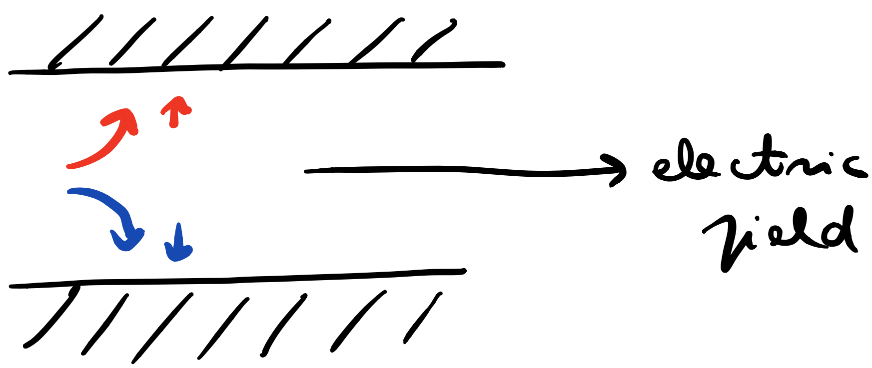
$$\sigma_{xy}^s \equiv \frac{j_x^s}{E_y}, \text{ where}$$

$$j_x^s = j_{x,\uparrow} - j_{x,\downarrow}$$
$$= \text{spin current along } x.$$

$$\Rightarrow \boxed{\sigma_{xy}^{s}} = \frac{j_{x,\uparrow}}{E_y} - \frac{j_{x,\downarrow}}{E_y}$$

$$= \sigma_{xy,\uparrow} - \sigma_{xy,\downarrow}$$

$$= \boxed{\frac{2e^2}{h}\,\mathrm{sgn}(\lambda_{so})}$$



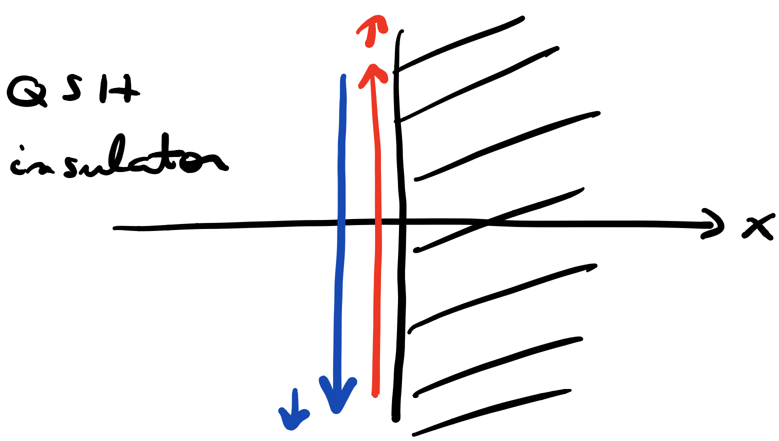
"spin accummulation" on edges.

$$\sigma_{xy}^{s} = \text{integer} \times \frac{e^2}{h}$$

"quantum" spin Hall insulator.

* Edge states:

Repeat the analysis of lecture 18 for each spin sector.



TRS ensures that ↑ and ↓ edge states counter-propagate.