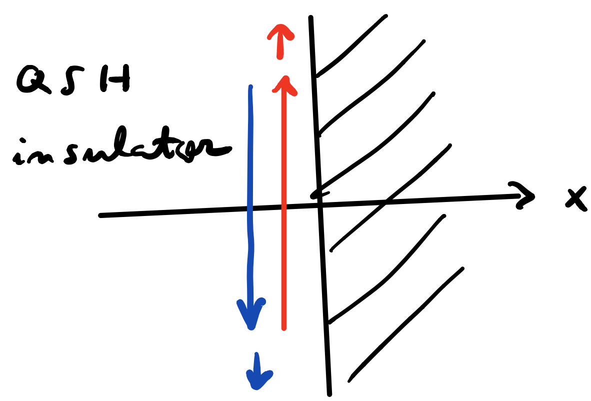
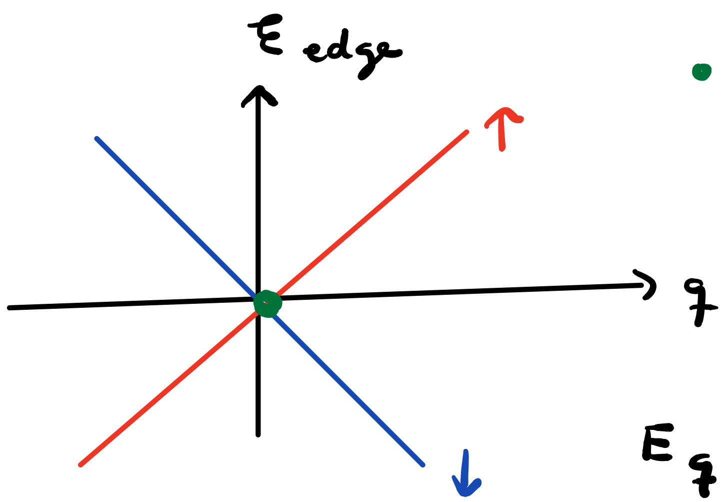
5 Quantum spin Hall insulator

5.1 Kane-Mele model

* Edge states:





- Kramers degeneracy at TRIM

$$E_{q\uparrow} = E_{-q\downarrow}$$

(TRS)

Two counter-propagating chiral edge states of opposite spin:

" helical edge states " "spin-momentum locking"

* Transport on the edge states is robust under perturbations that preserve TRS.

On a given edge, backscattering requires a spin-flip. This is not possible for non-magnetic impurities.

Let's show that

$$\langle \uparrow, -k \,|\, V \,|\, \downarrow, k \rangle = 0$$

if $[V, \Theta] = 0$.

$$|\downarrow, k\rangle = \Theta\, |\uparrow, -k\rangle \quad (1)$$

$$\Theta\, |\downarrow, k\rangle = \Theta^2\, |\uparrow, -k\rangle = -|\uparrow, -k\rangle \quad (2)$$

$\Theta^2 = -\mathbb{1}$

for spin $\frac{1}{2}$

$$\Rightarrow \langle \uparrow, -k | V | \downarrow, k\rangle =$$

$[V, \Theta] = 0$

$\Rightarrow \quad V = \Theta V \Theta^{-1}$

$$= \langle \uparrow, -k | \Theta V \Theta^{-1} | \downarrow, k\rangle$$

$$= \langle \uparrow, -k | \Theta V \, |\uparrow, -k\rangle$$

(1)

$\Theta^{-1}\Theta = \mathbb{1}$

$$= -\langle \downarrow, k |\, \overleftarrow{\Theta} \;\; \Theta \left(V \,|\, \uparrow, -k \rangle \right)$$

(2)

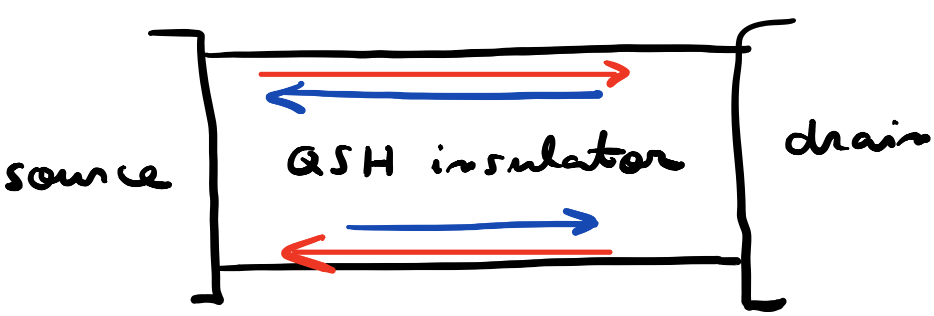
$$= -\langle \uparrow, -k \,|\, V \,|\, \downarrow, k \rangle$$

lecture 16 :

$$\left.\begin{aligned} |\tilde{\alpha}\rangle &= \Theta |\alpha\rangle \\ |\tilde{\beta}\rangle &= \Theta |\beta\rangle \end{aligned}\right\} \Rightarrow \langle \tilde{\beta} | \tilde{\alpha} \rangle = \langle \beta | \alpha \rangle^* = \langle \alpha | \beta \rangle$$

$$\Rightarrow \langle \uparrow, -k \,|\, V \,|\, \downarrow, k \rangle = 0 .$$

* Quantization of 2-terminal (charge) conductance



If interedge tunneling is negligible, both edges conduct in parallel.

If moreover magnetic scatterers are absent, each edge is free from backscattering.

Longitudinal charge conductivity:

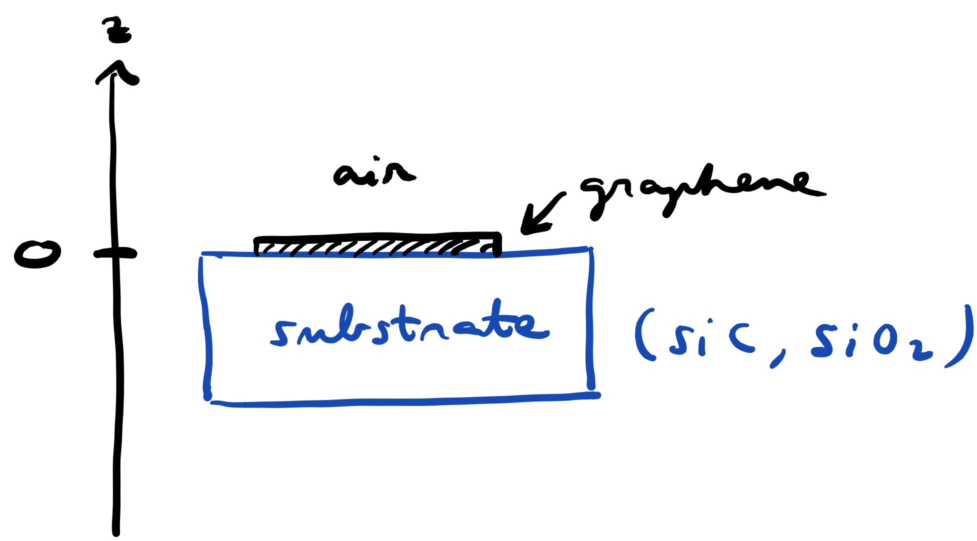
$$\sigma_L = \sigma_L^{(\text{top})} + \sigma_L^{(\text{bottom})}$$

$$= \frac{e^2}{h} + \frac{e^2}{h} = \frac{2e^2}{h}$$

Another reason why the Kane-Mele model is called "quantum" spin Hall insulator.

* Influence of spin mixing terms that preserve TRS?

In real 2D systems, spin-orbit terms that mix spin ↑ and spin ↓ are frequent (if not inevitable).
One such term is the "Rashba spin-orbit coupling"



$z \to -z$ symmetry is broken.

$$\Rightarrow \quad \underbrace{V(z)}_{} \neq V(-z)$$

potential that confines electrons in graphene to $z \simeq 0$.

$$\Rightarrow \quad \left.\frac{\partial V}{\partial z}\right|_{z=0} \neq 0$$

Relativistic effect: an e^- moving with velocity $\vec{v}$ in an electric field $\left.\frac{\partial V}{\partial z}\right|_{z=0} \hat{z}$ will experience a magnetic field

$$\vec{B} = \frac{\vec{v} \times \hat{z}}{c^2} \left.\frac{\partial V}{\partial z}\right|_{z=0}.$$

This B-field couples to the spin of the electron:

$$\boxed{\delta h_R} = \frac{g\,\mu_B}{2}\,\vec{B}\cdot\underbrace{\vec{\sigma}}_{\text{spin Pauli matrix}}$$

g-factor

$$= \boxed{\lambda_R\;\vec{\sigma}\cdot(\vec{p}\times\hat{z})}$$

where $\vec{p} = \underbrace{m}_{e^-\text{ mass}}\vec{v} = (p_x, p_y, 0)$

and $\lambda_R = \dfrac{g\,\mu_B}{2mc^2}\,\dfrac{\partial V}{\partial z}\Big|_{z=0}$

δh_R is the Rashba spin-orbit interaction.

$$[\delta h_R, \sigma^z] \neq 0$$

When $\delta h_R \neq 0$, we no longer have two independent copies of a Chern insulator.

Spin $\uparrow$ and spin $\downarrow$ are no longer good quantum #s $\Rightarrow$ spin Chern number is meaningless and σ_{xy}^s is no longer quantized.

<u>But</u>, the helical edge states (now without a well-defined spin direction) are still there and they are still robust (free from backscattering)

under non-magnetic perturbations.

$\Rightarrow$ quantization of 2-terminal longitudinal (charge) conductivity remains true.

The only way of removing the helical edge states is to have a large enough λ_R so that the bulk gap closes. Alternatively, one can remove the edge states by adding perturbations that break TRS.

The breakthrough of Kane and Mele was to identify a new topological invariant that explains the robustness of edge states.

5.2 $\mathbb{Z}_2$ topological invariant
(intuitive introduction)

* Consider the energy spectrum of an edge of a 2D insulator w/ TRS:

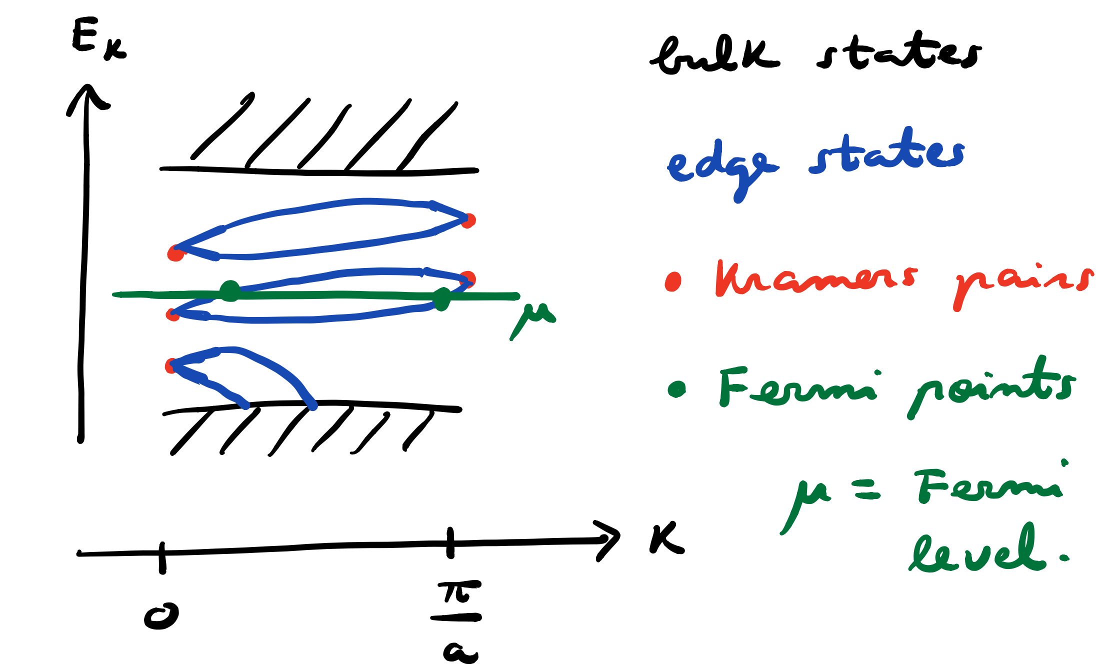
E_k, where k = momentum along edge (good quantum #)

* TRS $\Rightarrow$

$$\begin{cases} E_k = E_{-k} \quad (\text{focus on } k > 0) \\ \text{2-fold degeneracies at TRIM} \end{cases}$$

* Two types of energy spectra:

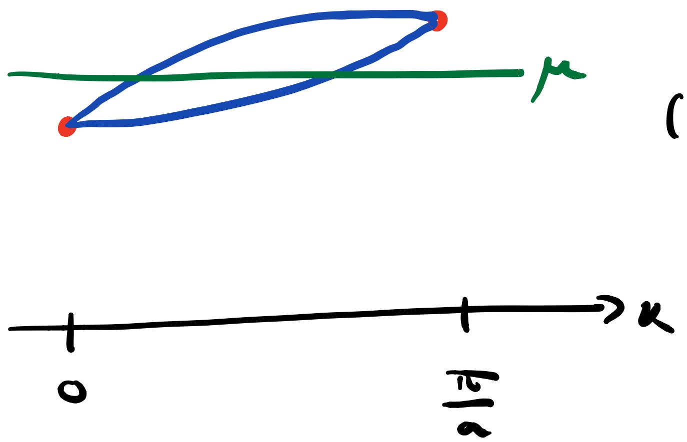
(i) Type 1:



Even # of Fermi points for $k \in [0, \frac{\pi}{a}]$ when the Fermi level is in the bulk gap.

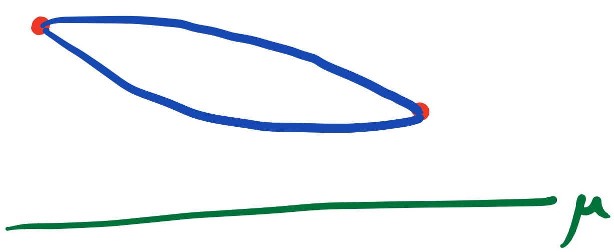
Edge is metallic for certain values of μ, but not for others.

Even when the edge is metallic, the metal is not robust.



(metallic edge)

$\downarrow$ smooth deformation which does not close bulk gap and preserves TRS

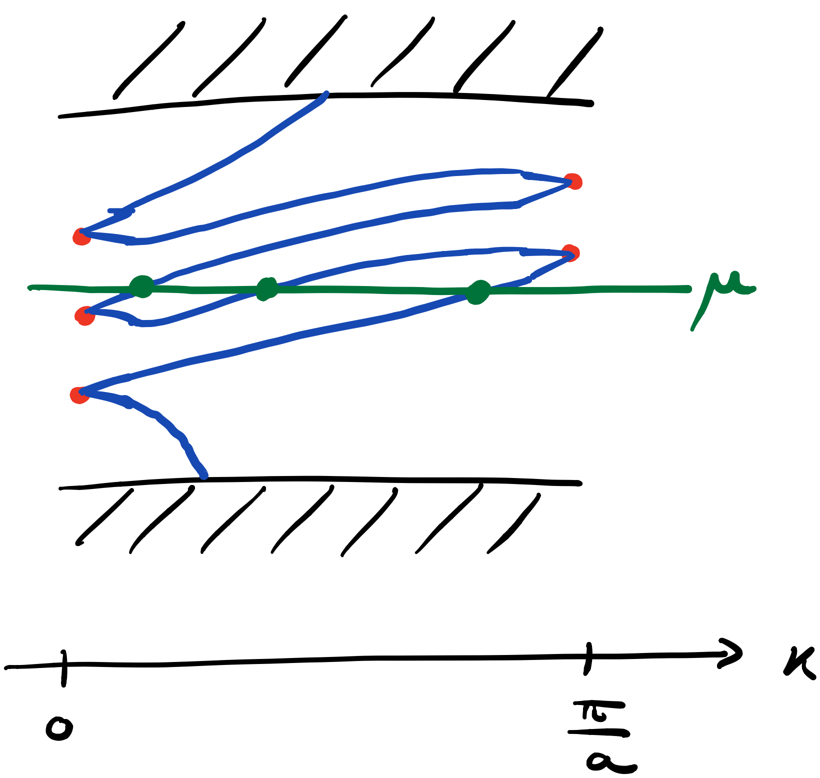


(insulating edge)

In sum: type 1 spectrum is smoothly connected to an insulating edge.

"topologically trivial insulator"

(ii) Type 2:



Odd # of Fermi points for $k \in [0, \frac{\pi}{a}]$ when the Fermi level is in the bulk gap.

Edge is always metallic when μ is inside the bulk gap.

Reason: edge bands connect bulk valence and conduction bands.

To go from type 1 to type 2 (or vice versa), one needs to break and then switch Kramers pairs.

This cannot be done smoothly if TRS is preserved.

$\Rightarrow$ type 1 and type 2 spectra are topologically distinct in the presence of TRS.

Type 2: "topological insulator" (protected by TRS)

* Topological invariant ?

ν = parity of # of Fermi points between $k=0$ and $k=\pi/a$

ν = even : trivial insulator

ν = odd : topological "

$\Rightarrow$ ν is a $\mathbb{Z}/2$ topological invariant.

In contrast, Chern # is a $\mathbb{Z}$ topological invariant.