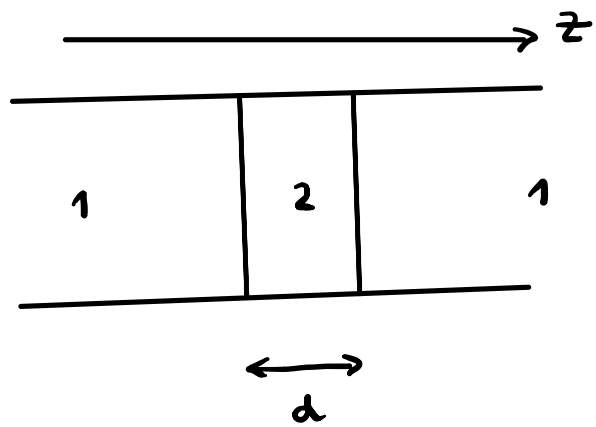
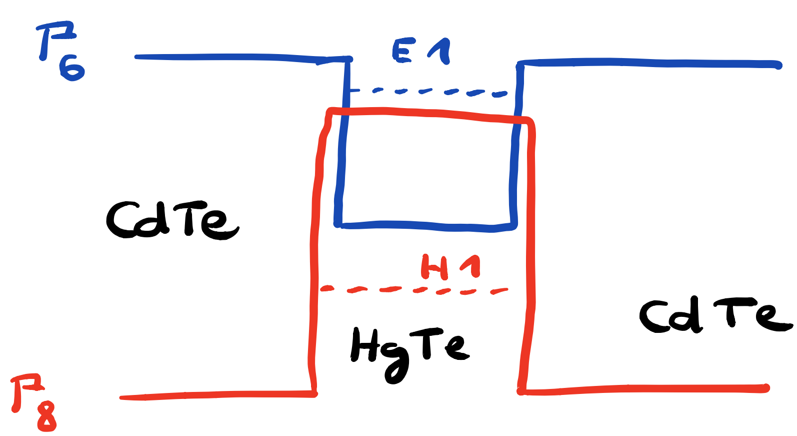
5 Quantum spin Hall insulator

5.6 BHZ model

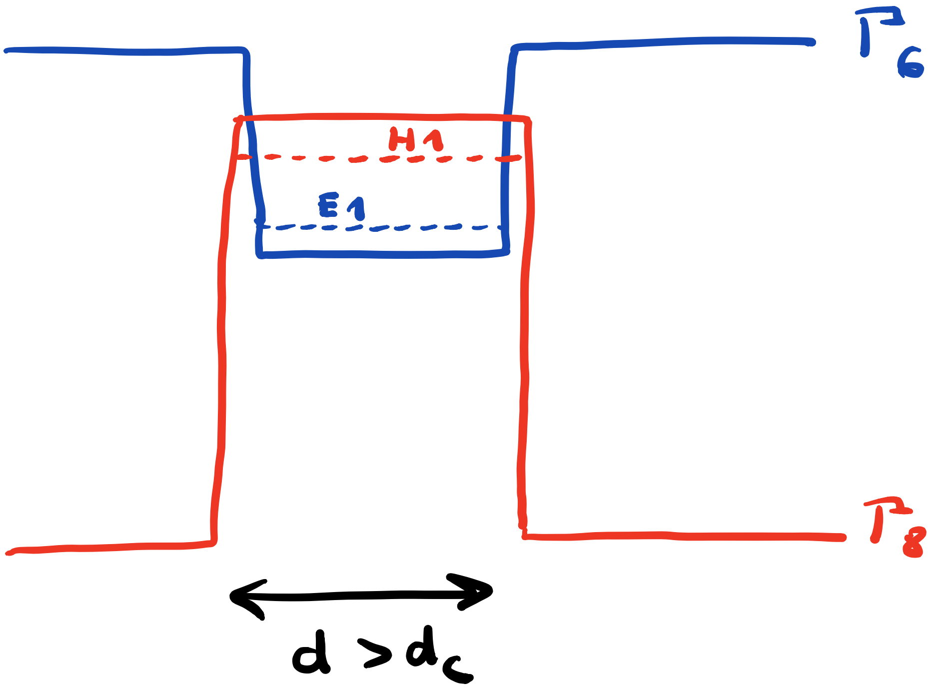


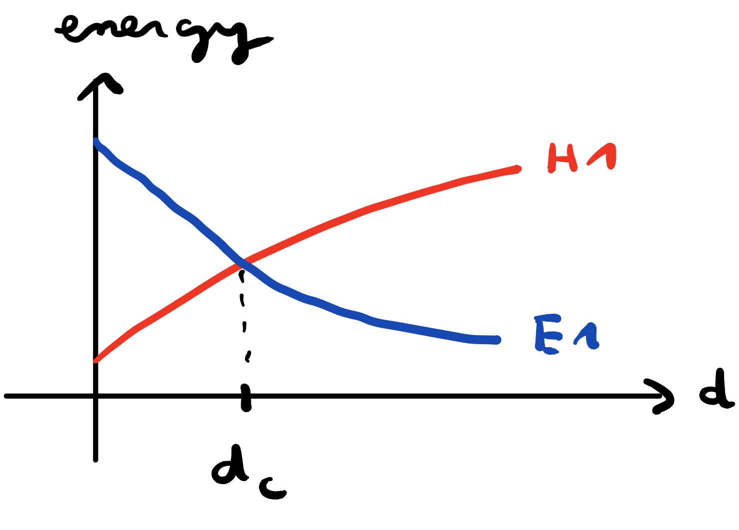
1 : CdTe (normal band ordering)

2 : HgTe (inverted " ")



$d < d_c$





$d_c = 6 \cdot 3$ nm

* Low-energy effective model :

$$h_{eff}(\vec{k})$$

Basis states (eigenstates at $\vec{k}=0$):

$$\{ |E1,+\rangle, |H1,+\rangle, |E1-\rangle, |H1-\rangle \}$$

$+,-$: Kramers partners

E, H : opposite parity

$$|E1,+\rangle \sim |s,\uparrow\rangle$$

$$|E1,-\rangle \sim |s,\downarrow\rangle$$

$$|H1,+\rangle \sim |p_x+ip_y,\uparrow\rangle$$

$$|H1,-\rangle \sim |p_x-ip_y,\downarrow\rangle$$

Inversion symmetry + rotational symmetry in the xy plane:

(i) $\langle E1, + | h_{eff} | H1, + \rangle$

$\sim k_x + i k_y$

(ii) $\langle E1, - | h_{eff} | H1, - \rangle$

$\sim k_x - i k_y$

(iii) $\langle \alpha, + | h_{eff} | \beta, - \rangle = 0$

(iv) $\langle E1, \pm | h_{eff} | E1, \pm \rangle$

and $\langle H1, \pm | h_{eff} | H1, \pm \rangle$

even in $\vec{k}$

From these considerations:

2x2 matrix

$$h_{eff}(\vec{k}) = \begin{pmatrix} h(\vec{k}) & 0 \\ 0 & h(-\vec{k})^* \end{pmatrix}$$

no matrix elements between + and - (due to SIS + TRS)

where

$$h(\vec{k}) = \epsilon(\vec{k})\mathbb{1} + \vec{d}(\vec{k})\cdot\vec{\sigma}$$

$\sigma^z \uparrow$: E

$\sigma^z \downarrow$: H

$$\epsilon(\vec{k}) = C + D\left(k_x^2 + k_y^2\right)$$

$$d_x(\vec{k}) = A k_x$$

$$d_y(\vec{k}) = -A k_y$$

$$d_z(\vec{k}) = M - B(k_x^2 + k_y^2)$$

A, B, C, D, M : material parameters that depend also on d.

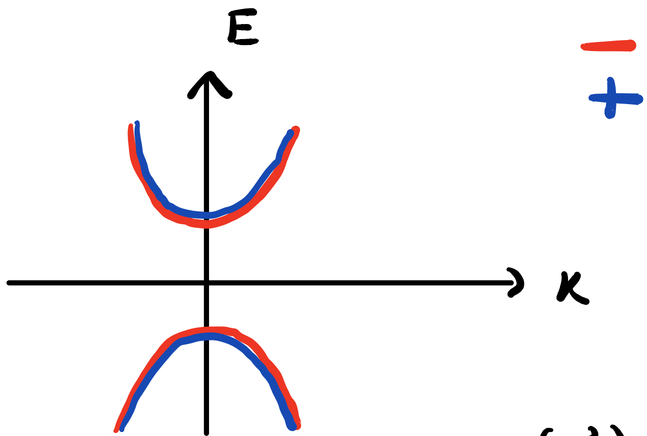
It turns out $B < 0$.

"BHZ model".

$h(\vec{k})$ is a 2D Dirac Hamiltonian, w/ Dirac mass

$$d_z(\vec{k}) \equiv M(\vec{k})$$

Energy spectrum:



$h(\vec{k}) \rightarrow +$

$h(-\vec{k})^* \rightarrow -$

* Symmetries of BHZ model:

(1) TRS

$\Theta = -i\,\tau^y\,\underline{K}$

τ^z : $+, -$ pseudospin

Θ exchanges $+ \leftrightarrow -$

$$h_{eff}(\vec{k}) = \Theta \, h_{eff}(-\vec{k}) \, \Theta^{-1}$$

(2) SIS

$$\Pi = \sigma^z$$

$$\Pi | E, \cdots \rangle = | E, \cdots \rangle$$

$$\Pi | H, \cdots \rangle = - | H, \cdots \rangle$$

$$h_{eff}(\vec{k}) = \Pi \, h_{eff}(-\vec{k}) \, \Pi^{-1}$$

Because SIS + TRS, bands are 2 fold degenerate at all $\vec{k}$.

* Remarks :

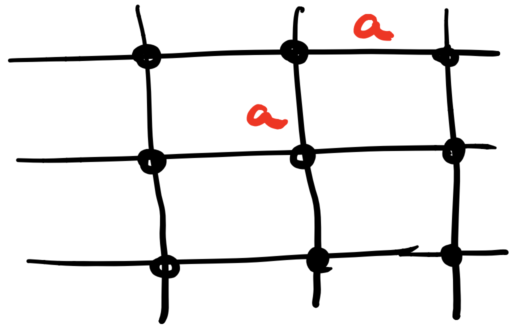
(1) Dirac mass at $k = 0$, M, is the energy difference

between E1 and H1 states.

If E1 and H1 get inverted as a function of d, then M changes sign.

If M changes sign at an interface, we anticipate the appearance of gapless edge states (generalisation of Jackiw-Rebbi modes to 2D).

* Lattice regularisation. Useful to compute topological invariants.



$$h_{eff}(\vec{k}) = \begin{pmatrix} h(\vec{k}) & 0 \\ 0 & h(-\vec{k})^* \end{pmatrix}$$

$$h(\vec{k}) = \epsilon(\vec{k}) + \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

$$\epsilon(\vec{k}) = C - \frac{2D}{a^2}\left[2 - \cos(k_x a) - \cos(k_y a)\right]$$

$$d_x(\vec{k}) = \frac{A}{a} \sin(k_x a)$$

$$d_y(\vec{k}) = \frac{-A}{a} \sin(k_y a)$$

$$d_z(\vec{k}) = M - 2\frac{B}{a^2}\left[2 - \cos(k_x a) - \cos(k_y a)\right]$$

Trick:

continuum model	lattice model
k_x	$\frac{1}{a}\sin(k_x a)$
k_y	$\frac{1}{a}\sin(k_y a)$
k_x^2	$\frac{2}{a^2}\left(1-\cos(k_x a)\right)$
k_y^2	$\frac{2}{a^2}\left(1-\cos(k_y a)\right)$

When $a \to 0$, or $\vec{k} \to 0$

lattice model $\to$ continuum model.

* Topological phase diagram

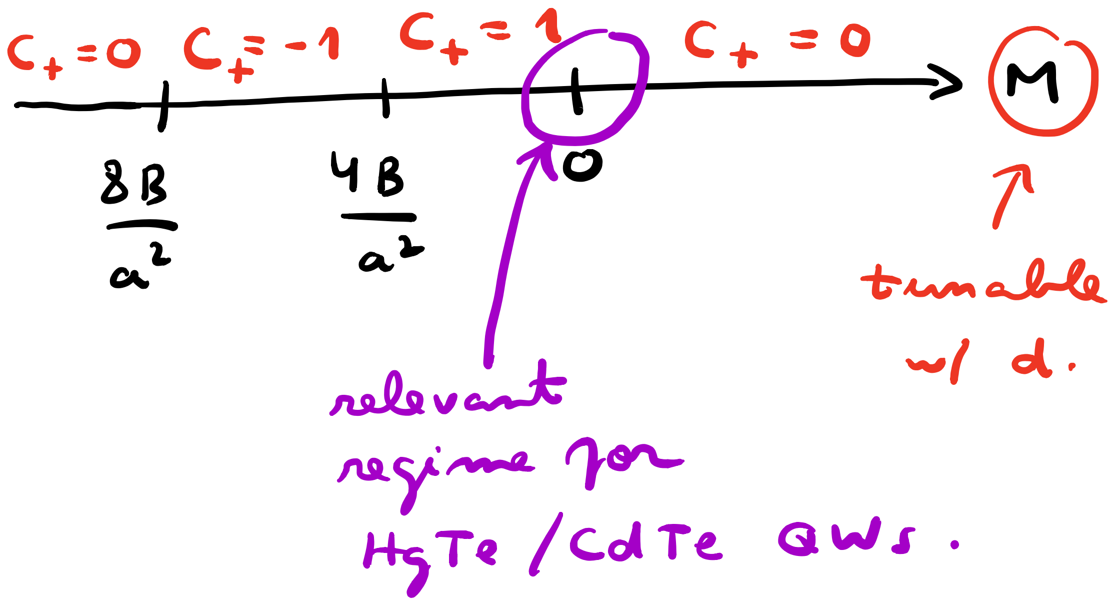
1) Chern # for occupied bands:

C_+ , C_-

$$C_n = \frac{1}{4\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_x \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_y$$

$$\hat{d}_{\vec{k},n} \cdot \left(\frac{\partial \hat{d}_{\vec{k},n}}{\partial k_x} \times \frac{\partial \hat{d}_{\vec{k},n}}{\partial k_y} \right)$$

Same calculation as in problem # 1 of hw # 4.



$$C_- = -C_+ \quad (\text{TRS})$$

$$\Rightarrow \sigma_{xy} = 0.$$

$C_+ \neq 0$ when $\hat{d}$ covers the full sphere when $\vec{k}$ is varied in the 1BZ.

2) $\mathbb{Z}_2$ invariant from the Fu-Kane formula

$$(-1)^{\nu} = \prod_i \delta_i \quad , \quad i \in \text{TRIM}$$

$$\delta_i = \prod_{m=1}^{N} \xi_{2m}(\Gamma_i)$$

2N occupied bands

$m \in$ occupied bands

(i) TRIM:

$$\Gamma_1 = (0,0)$$

$$\Gamma_2 = \frac{\pi}{a}(1,0)$$

$$\Gamma_3 = \frac{\pi}{a}(0,1)$$

$$\Gamma_4 = \frac{\pi}{a}(1,1)$$

(ii) Occupied bands:

one + band and one − band.

$N = 1$ for the BHZ model.

we need the parity

eigenvalue of only one band.

Let's consider just the + band.

(iii) Calculation of parity

eigenvalues:

- $\Gamma_1 = (0,0)$

$$\underbrace{h(\Gamma_1)}_{+\text{ states.}} = C + M \sigma^z$$

E, H pseudospin

occupied "+" state:

$|E1, +\rangle$ if $M < 0$

$|H1, +\rangle$ if $M > 0$

$\Rightarrow$ parity of occupied "+" state

$$\underset{\substack{\uparrow \\ \Pi|E\rangle = |E\rangle \\ \Pi|H\rangle = -|H\rangle}}{=} - \operatorname{sgn}(M) \equiv \zeta_+(\Gamma_1)$$

- $\Gamma_2 = \left(\frac{\pi}{a}, 0\right)$

$$\underbrace{h(\Gamma_2)}_{\substack{\uparrow \\ +\text{ states}}} = C - \frac{4D}{a^2} + \left(M - \frac{4B}{a^2}\right)\sigma^z$$

occupied "+" state:

$|E1,+\rangle$ if $M - \frac{4B}{a^2} < 0$

$|H1,+\rangle$ if $M - \frac{4B}{a^2} > 0$

Parity eigenvalue for the occupied "+" band: $\xi_+(\Gamma_2)$

$$= -\mathrm{sgn}\left(M - \frac{4B}{a^2}\right)$$

- $\Gamma_3 = \left(0, \frac{\pi}{a}\right)$

$$\xi_+(\Gamma_3) = -\mathrm{sgn}\left(M - \frac{4B}{a^2}\right)$$

- $\Gamma_4 = \frac{\pi}{a}(1,1)$

$$h(\Gamma_4) = C - 8\frac{D}{a^2} + \left(M - \frac{8B}{a^2}\right)\sigma^z$$

occupied "+" state:

$|E1,+\rangle$ if $M - \frac{8B}{a^2} < 0$

$|H1,+\rangle$ if $M - \frac{8B}{a^2} > 0$

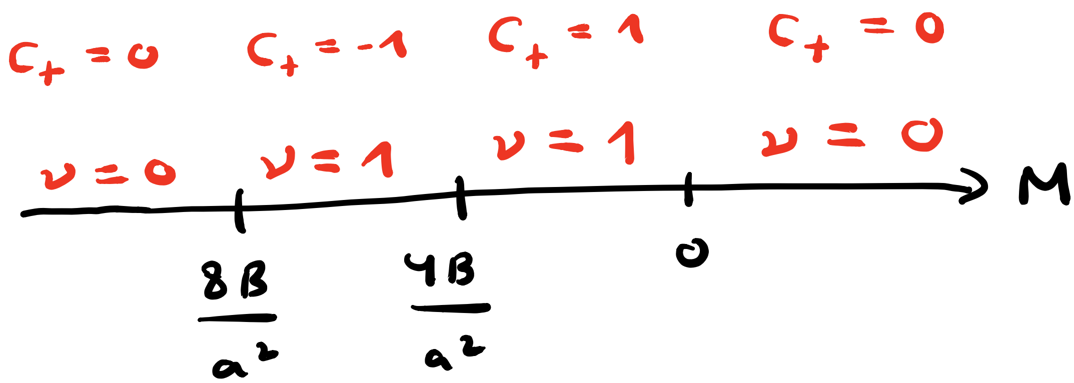
$$\xi_+(\Gamma_4) = -\,\mathrm{sgn}\left(M - \frac{8B}{a^2}\right)$$

(iv) Then,

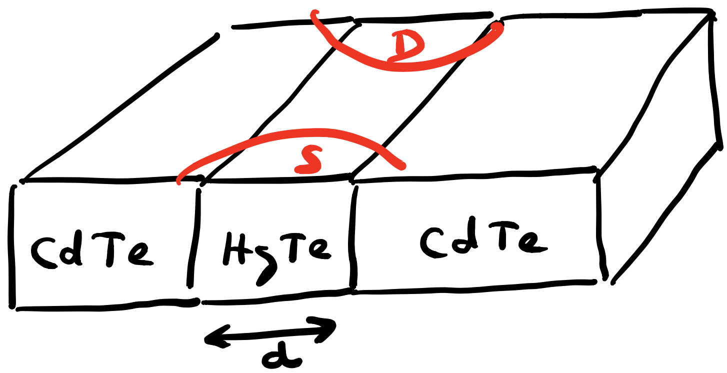
$$\boxed{(-1)^\nu} = \xi_+(\Gamma_1)\,\xi_+(\Gamma_2)\,\xi_+(\Gamma_3)\,\xi_+(\Gamma_4)$$

$$= \operatorname{sgn}(M)\left[\operatorname{sgn}\left(M - \frac{4B}{a^2}\right)\right]^2 \operatorname{sgn}\left(M - \frac{8B}{a^2}\right)$$

$$= \boxed{\operatorname{sgn}(M)\operatorname{sgn}\left(M - \frac{8B}{a^2}\right)}$$



5.7 Experimental discovery

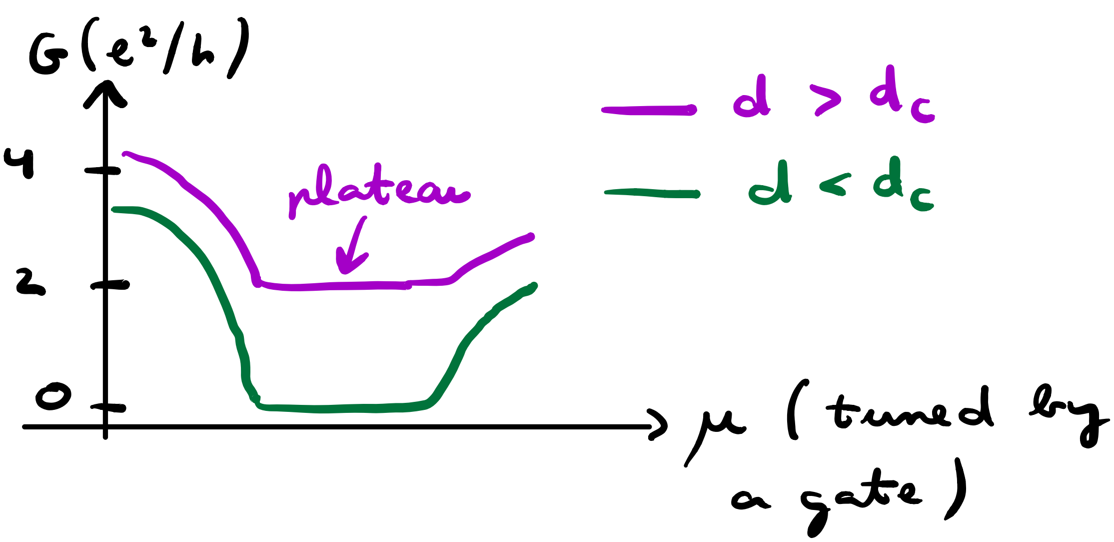


S : source

D : drain

Source-drain conductance :

$$G = \frac{I}{V}$$



$\longleftrightarrow$

bulk gap

$\frac{2e^2}{h}$ plateau in conductance coming from helical edge states free from backscattering.

* Drawbacks: HgTe / CdTe QWs are difficult to realise. Also, the bulk gap is small (~ 10 meV) $\Rightarrow$ quantization of G requires $T < 10$ K.

* Search for other QSH insulators. Candidates: 2D germanene films, BiF_2 layers, stanene, ...

still ongoing.