End-of-semester plan:

1) No more official homeworks.
   There will be an optional one.

2) Lectures will continue until
   April 30th.

3) Final presentations:
   April 29th (afternoon)
   April 30th (morning)

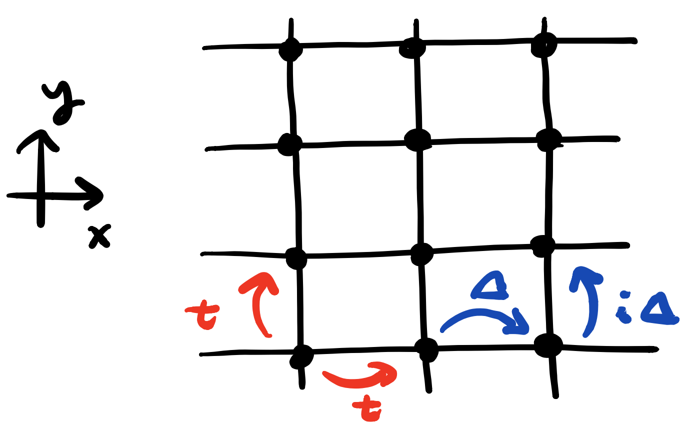
⑥ Chiral topological superconductors

lecture 10: 1D topological SC.
(Kitaev chain)

Can we build a 2D topological SC by stacking Kitaev chains? yes.
"ladder construction"

(this construction can also be used to create a QAH insulator by stacking SSH chains)

* 2D lattice of spinless fermions



$a \equiv 1$

t = single-particle hopping

Δ = pairing amplitude

* Grand canonical Hamiltonian :

$$\mathcal{H} = \sum_{\vec{k}} \left(c^{+}_{\vec{k}},\, c_{-\vec{k}} \right) h(\vec{k}) \begin{pmatrix} c_{\vec{k}} \\ c^{+}_{-\vec{k}} \end{pmatrix}$$

$$\vec{k} = (k_x, k_y)$$

$$h(\vec{k}) = \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

$\sigma^z \uparrow$: electron

$\sigma^z \downarrow$: hole

$$d_x(\vec{k}) = -\Delta \sin k_y$$

$$d_y(\vec{k}) = -\Delta \sin k_x$$

$$d_z(\vec{k}) = \frac{1}{2} \xi_{\vec{k}}$$

$$\xi_{\vec{k}} = -2t\,(\cos k_x + \cos k_y) - \underbrace{\mu}_{\text{chemical potential}}$$

$$h(\vec{k}) = \begin{pmatrix} \frac{\xi_{\vec{k}}}{2} & \underbrace{-\Delta \sin k_y + i\Delta \sin k_x}_{\text{pairing}} \\ c.c. & -\frac{\xi_{\vec{k}}}{2} \end{pmatrix}$$

odd-parity pairing

$p_x + i p_y$ pairing

* $h(\vec{k})$ is formally very similar to the Hamiltonian of a Chern

insulator in the square lattice
(hw # 4).

* Broken time-reversal symmetry.

$\Theta = \underline{K}$ (complex conjugation)

spinless fermions

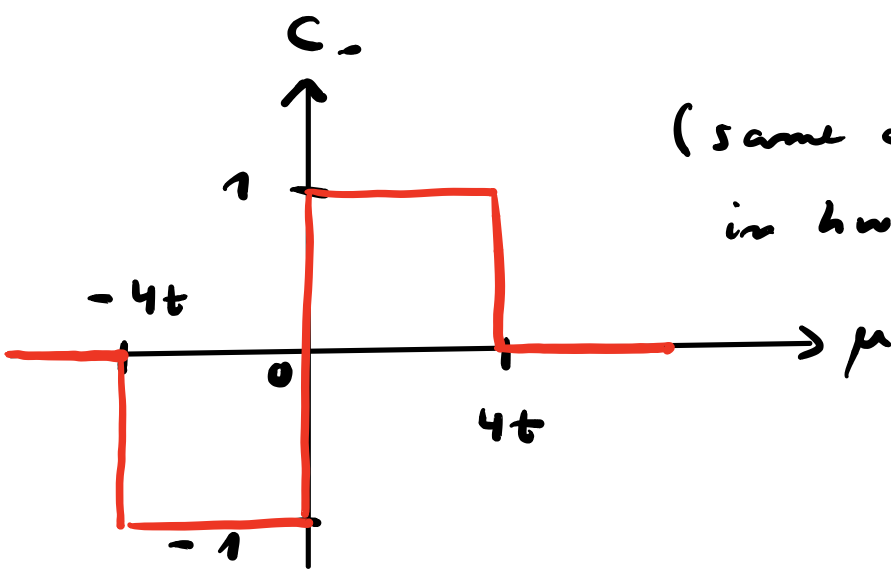
TRS $\Leftrightarrow$ $h(\vec{k}) = \Theta\, h(-\vec{k})\, \Theta^{-1}$

$$= h(-\vec{k})^*$$

This relation is not satisfied.

6.1 Topological phase diagram

C_- = Chern # of occupied band

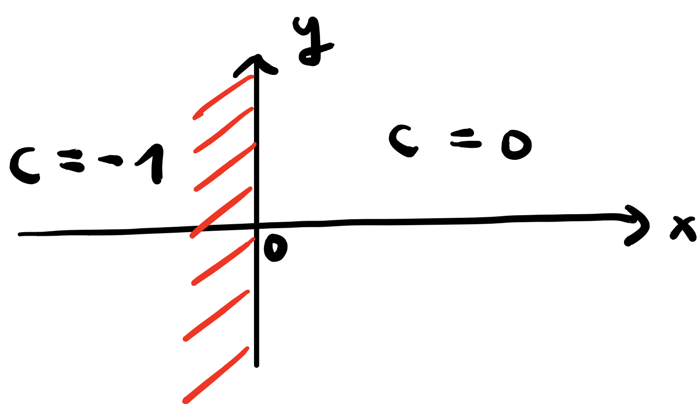


(same as in hw # 4)

$\mu \in (-4t, 4t) \to$ topological SC

$|\mu| > 4t \to$ trivial SC

6.2 Chiral Majorana edge states



Suppose $\mu \simeq -4t$

Then, the low-energy electronic states are located near

$$(k_x, k_y) = (0, 0).$$

Near this point,

$$h(\vec{q}) \approx -\Delta \sigma^y q_x - \Delta \sigma^x q_y + m \sigma^z$$

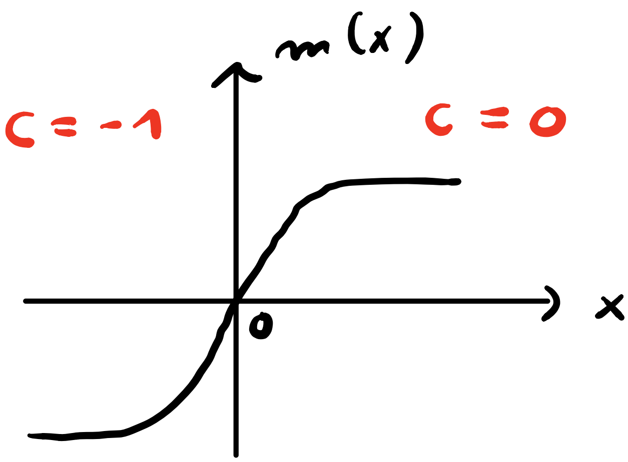
measured from (0,0)

(2D Dirac fermions)

where m = Dirac mass

$$\equiv -4t-\mu$$

Domain wall at $x=0$:



$$h(q) = \Delta \sigma^z i\partial_x - \Delta q \sigma^x$$

$$+ m(x)\sigma^z$$

momentum along the edge (along y)

At $q=0$, 1D Dirac Hamiltonian with a domain wall in the mass $\rightarrow$ zero-energy mode localised at $x=0$:

$$\phi_{edge}(x) \propto \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} e^{-\frac{1}{v}\int_0^x m(x')dx'}$$

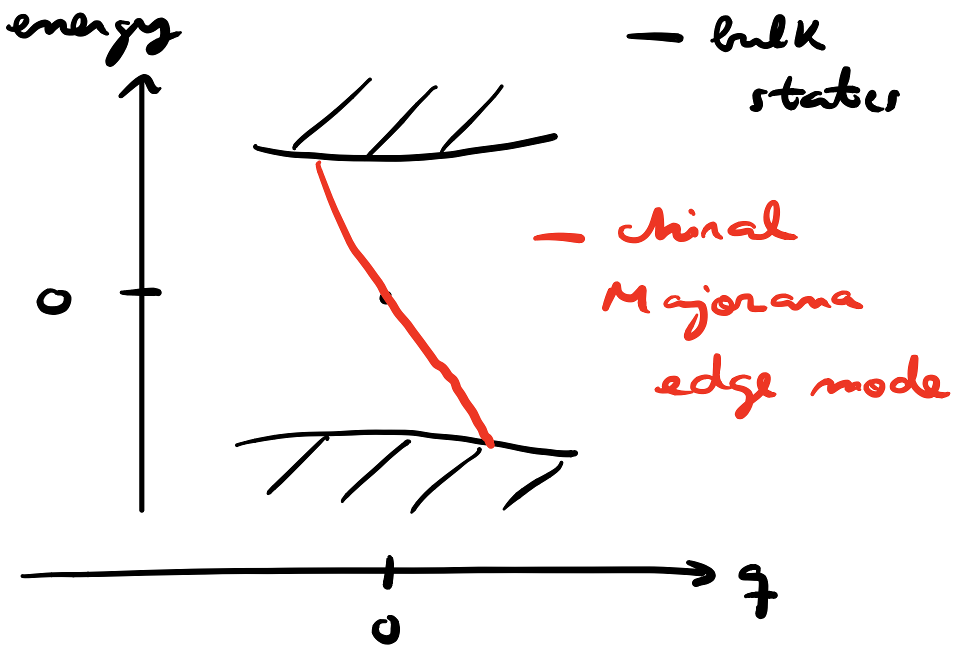
equal superposition of electron and hole

$\rightarrow$ Majorana zero mode.

At $q \neq 0$,

$$\psi_{edge}(x,y) = e^{iqy}\,\phi_{edge}(x)$$

with $E_{edge}(q) = -\Delta q$



6.3 Quantized thermal Hall effect

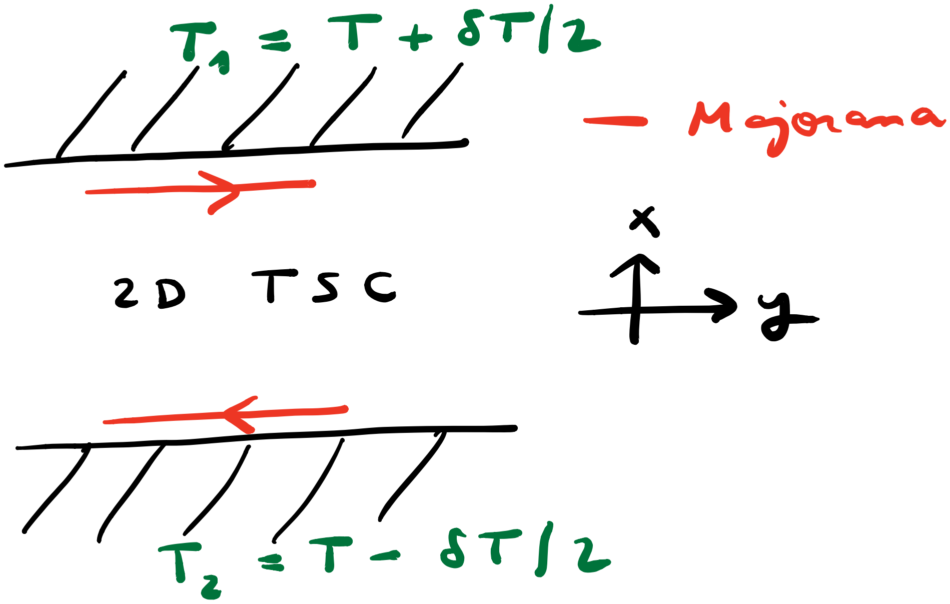
* In QAH insulators,

ν chiral edge states $\rightarrow$ $\sigma_{xy} = \frac{e^2}{h}\nu$

Counterpart of this in a 2D chiral superconductor?

* Majoranas are charge neutral
  → they carry no electric current.

But, they can carry <u>heat</u>.



Temperature gradient along x.

Heat current density along y:

$$j_{Q,y} = \underbrace{j_{Q,y}^{\text{top}}}_{\text{top Majorana contribution}} + j_{Q,y}^{\text{bottom}}$$

Only chiral Majorana modes carry heat at temperature $\ll$ bulk gap. The contribution of bulk states is exponentially suppressed.

$$= \frac{1}{L} \sum_k v_k E_k \underbrace{f_k(T_1)}_{\text{Fermi-Dirac distribution, temp. } T_1}$$

system size along y

momentum along edge

$$- \frac{1}{L} \sum_k v_k E_k f_k(T_2)$$

$$= \frac{1}{L} \sum_k v_k E_k \left[f_k(T_1) - f_k(T_2) \right]$$

$$= \int \frac{dk}{2\pi} \frac{\partial E_k}{\hbar \partial k} E_k \, \delta T \underbrace{\frac{\partial f_k}{\partial T}}_{-\frac{E_k}{T} \frac{\partial f_k}{\partial E_k}}$$

δT small

$$= -\frac{1}{2\pi\hbar} \frac{\delta T}{T} \int_{-\infty}^{\infty} dE \, E^2 \frac{\partial f(E)}{\partial E}$$

$$= \frac{1}{2\pi\hbar} \frac{\delta T}{T} \frac{1}{3} k_B^2 \pi^2 T^2$$

$$= \frac{\pi k_B^2}{6\hbar} T \, \delta T$$

* Thermal Hall conductivity:

$$\boxed{\kappa_{xy} = \frac{\dot{j}_{Q,y}}{\delta T} = \frac{\pi k_B^2}{6\hbar} T}$$

universal #

This is applicable to QAH insulators.

Where does the Majorana "character" of the edge states appear?

A Majorana is a "half-fermion":

$$\underset{\uparrow\ \text{full fermion}}{c} = \underbrace{\gamma_1}_{} + i \underbrace{\gamma_2}_{\text{two Majoranas}} \qquad \text{(lec. 10)}$$

Therefore, it turns out that the above expression for κ_{xy} has to be divided by 2 in the case of chiral Majorana modes.

In sum:

ν chiral Majorana edge states

$$\rightarrow \boxed{\kappa_{xy} = \nu \frac{\pi k_B^2}{12 \hbar} T}$$

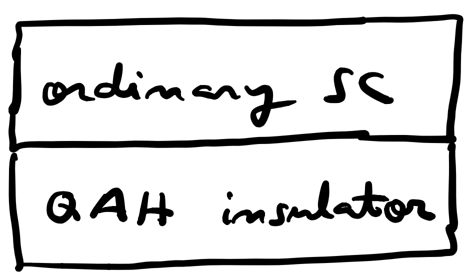
Remark: we have neglected contribution from phonons.

6.4 Experimental realisation

* Moore - Read state of a fractional quantum Hall insulator (filling factor $\nu = 5/2$)

[N. Read and D. Green, PRB 61, 10267 (2000)]

*



Q. L. He et al., Science 357, 294 (2017).

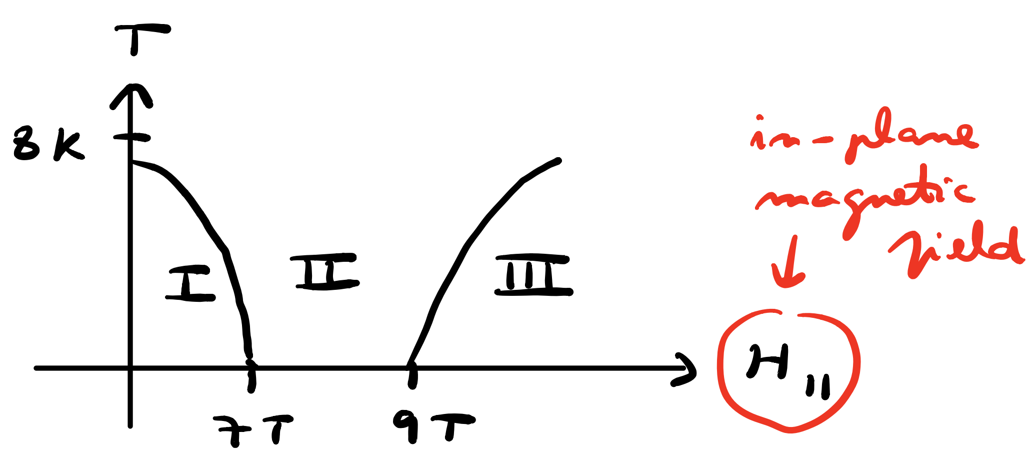
Possible application for topological quantum computation:

B. Lian et al, PNAS 23, 115 (2018)

(see also the 2018 APS MM talk by S.C. Zhang, on youtube. Google "discovery of chiral Majorana")

* Spin liquids ($\alpha - RuCl_3$)

Phase diagram:



I : Antiferromagnetic insulator (topologically trivial)

III : spin liquid (topologically trivial)

II : Kitaev spin liquid (topologically nontrivial w/ chiral Majorana edge states)

In region II, quantization of κ_{xy} has been reported at low temperature.

Y. Kasahara et al., Nature 559, 227 (2018).

(7) Helical topological SCs

Spinful electrons in 2D.

The most basic realisation:

2 copies of chiral topological SC,

$p_x + ip_y$ for spin $\uparrow$ electrons,

$p_x - ip_y$ for spin $\downarrow$ electrons

(so as to preserve time-reversal symmetry).

$$\mathcal{H} = \sum_{\vec{k}} \left(c^{\dagger}_{\vec{k}\uparrow} \; c_{-k\uparrow} \; c^{\dagger}_{k\downarrow} \; c_{-k\downarrow} \right) h(\vec{k}) \begin{pmatrix} c_{\vec{k}\uparrow} \\ c^{\dagger}_{-\vec{k}\uparrow} \\ c_{\vec{k}\downarrow} \\ c^{\dagger}_{-\vec{k}\downarrow} \end{pmatrix}$$

where

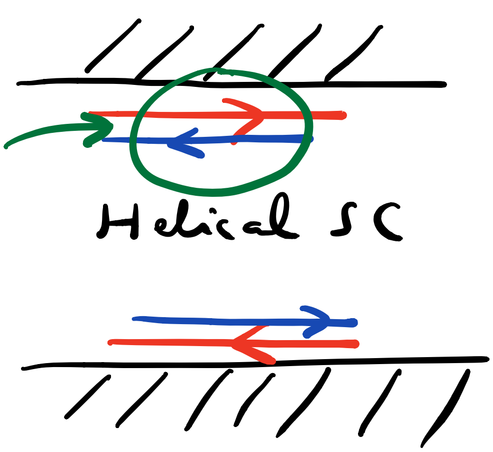
$$h(\vec{k}) = \begin{pmatrix} h_{chiral}(\vec{k}) & 0 \\ 0 & h_{chiral}(-\vec{k})^* \end{pmatrix}$$

and $h_{chiral}(\vec{k})$ is the same as in Sec. 6.

This has the same form as the BHZ Hamiltonian.

* Helical Majorana edge modes

one helical edge states

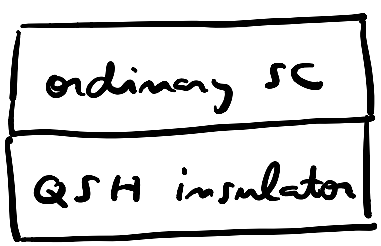


Red and blue are time-reversed partners.

In the presence of spin mixing terms, an odd # of helical edge states per edge is still robust. (same reason as in lecture 21).

The topological invariant is $\mathbb{Z}/_2$.

* Experimental reports:



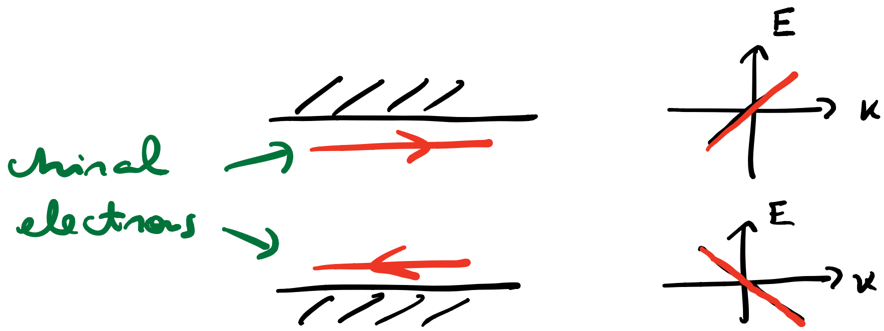
S. Hart et al., Nature Physics 10, 638 (2014).

Helical Majorana edge modes have not been discovered yet.

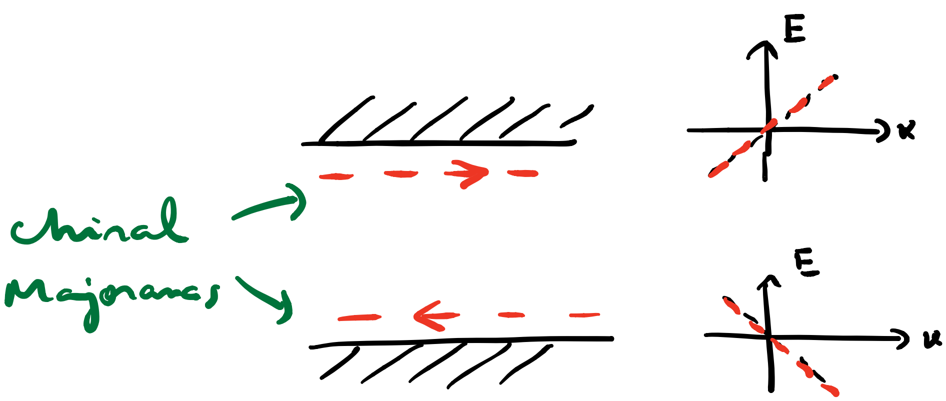
(8) Summary: band topology in 2D

(1) Broken TR ($\mathbb{Z}$ topological invariant)

(i) QAH insulator

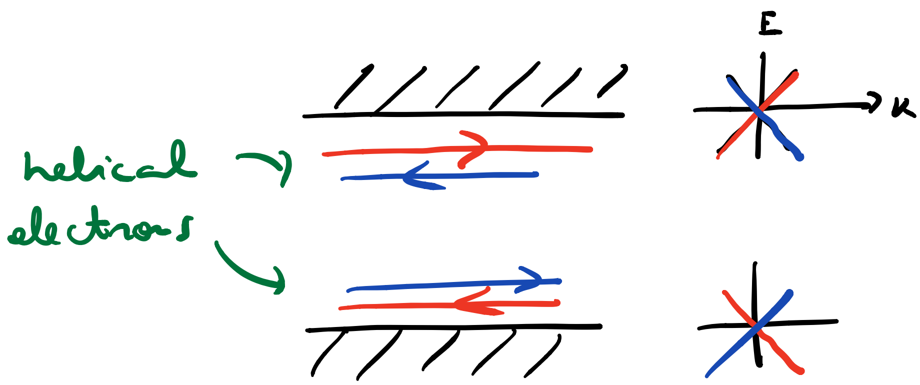


(ii) Chiral SC



(2) Unbroken TR ($\mathbb{Z}_2$ invariant)

(i) QSH insulator



(ii) Helical SC