② Thouless charge pump

1D insulator:

Time-periodic perturbation.

$$\underbrace{\mathcal{H}(k,t)}_{\text{periodic in } k \text{ and in } t} |u_{kn}(t)\rangle = E_{kn}(t) |u_{kn}(t)\rangle$$

Charge transported in one cycle of the perturbation:

$$e \sum_{n \in occ} C_n$$

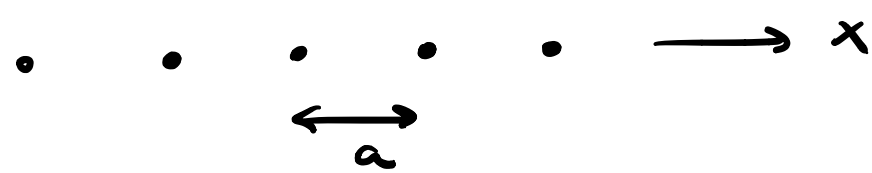
Chern # for band n

$$C_n = \frac{1}{2\pi} \int_0^T dt \int_{-\frac{\pi}{a}}^{\pi/a} dk \; B_n(k,t)$$

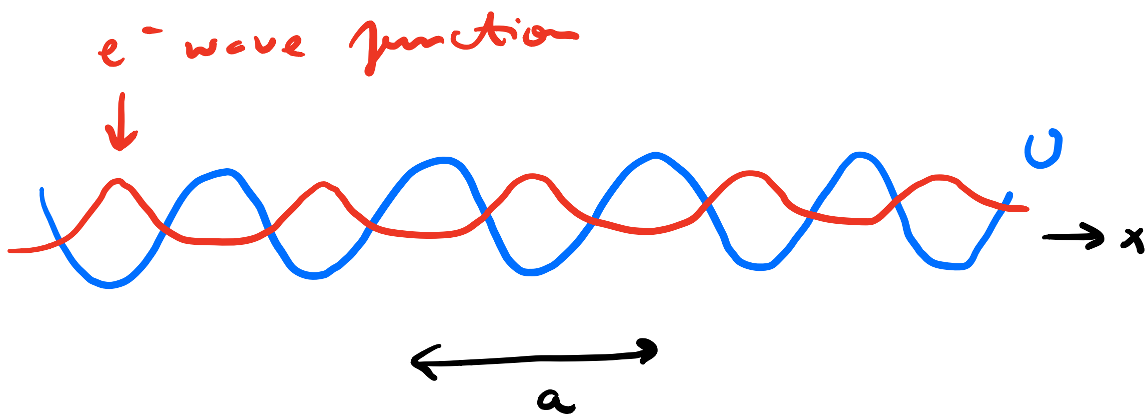
Closed surface : 2D Brillouin zone in (k, t) space.

2.1 Example

Spinless e^--s in a 1D lattice.



$$U(x) = U(x+a)$$



Define n = density of e^--s
(# of e^--s per unit length)

Then, na = # of e^--s in one unit cell

= # of filled bands.

Justification:

# of filled bands =

$$= \frac{\text{\# of } e^-\text{-s in the crystal}}{\text{\# of } e^-\text{-s per filled band}}$$

$$= \frac{n\,L}{\dfrac{\frac{2\pi}{a}}{\frac{2\pi}{L}}}$$

L ← length of crystal

$\frac{2\pi}{a}$ ← width of BZ

$\frac{2\pi}{L}$ ← distance between neighboring k-points

$= n a$

In an insulator,

$$n a = N \in \mathbb{N}$$

Now let's make the potential time-dependent:

$$U(x) \longrightarrow U(x,t) = U(x - vt)$$

$v =$ constant

"travelling wave (or sliding) potential"

For a fixed x, the potential varies periodically in time, with a period

$$T = \frac{a}{v}$$

If electrons follow the sliding potential adiabatically, there will be a current

$$I = e\, n\, v \quad \left(= \frac{e n a}{T} \right)$$

DC current produced by an AC perturbation.

No dissipation associated to the current (the system is always at its instantaneous ground state)

Charge pumped over one cycle:

$$IT = e\, n\, v\, \frac{a}{v} =$$

$$= e\, n\, a$$

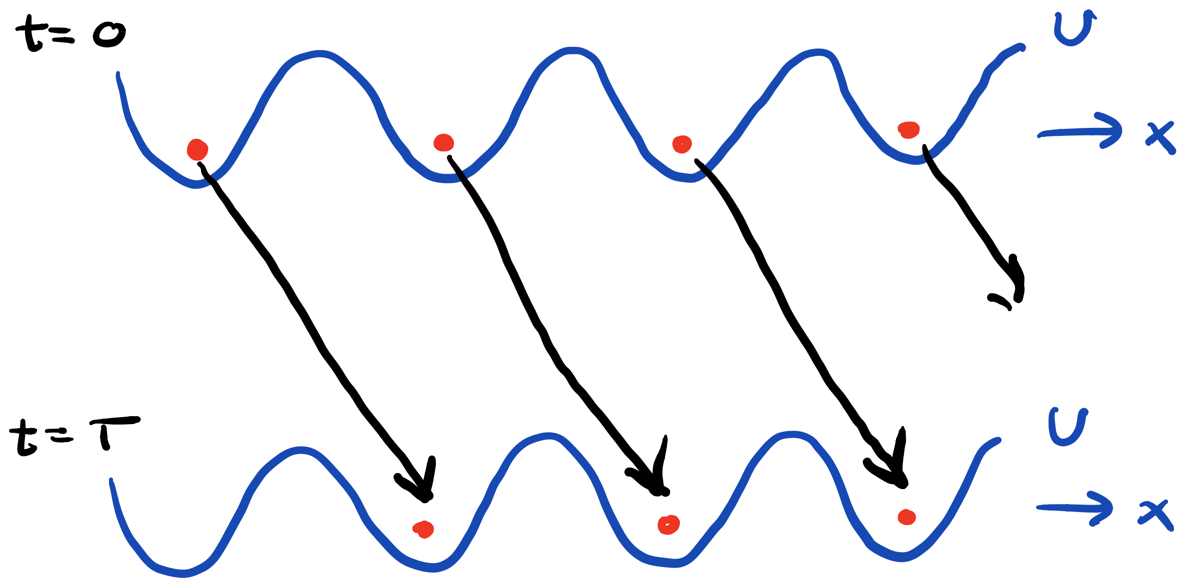
$$= e\, \underbrace{N}_{\substack{\uparrow \\ \text{\# of occupied} \\ \text{bands (= integer)}}}$$

General formula:

$$\text{Pumped charge} = e \sum_{n \in occ} C_n$$

Then, in this example,

$C_n = 1$ for all occupied bands.



Very similar to last lecture's picture for the case $C_n = 1$, in terms of Wannier centers.

Classical counterpart:

Archimedes' screw.

Inverse classical effect:

If water is fed on top of an

Archimedes' screw, it will force the screw to rotate

$\longrightarrow$ electric generator.

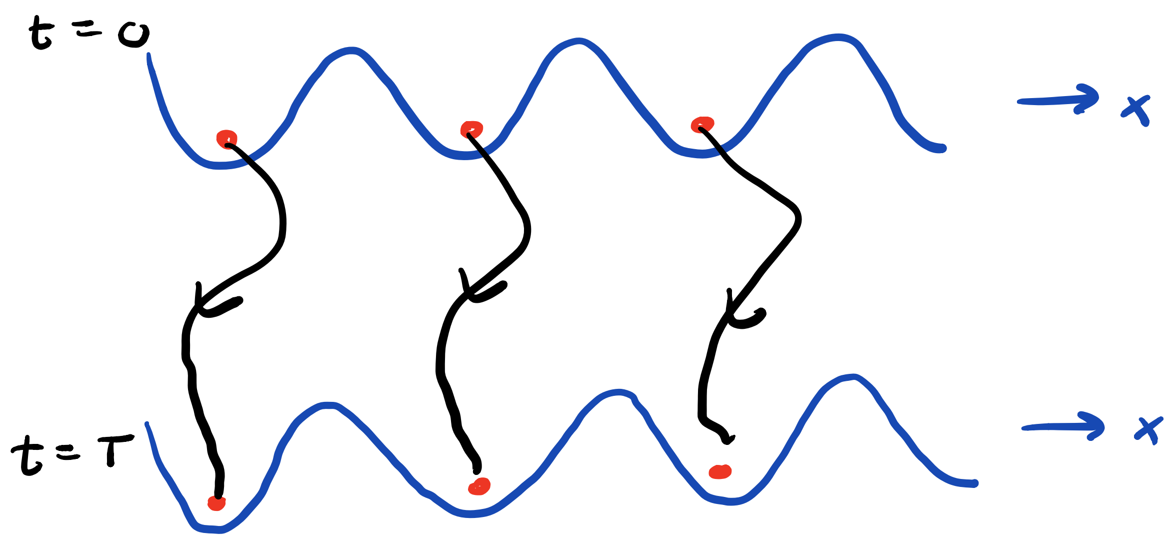
What is the quantum analogue of the inverse classical effect?

* Not all time-periodic potentials are able to pump charge.

Example :

$$U(x,t) = \underbrace{U(x)}_{U(x) = U(x+a)} \cos\left(\frac{2\pi}{T} t\right)$$

"standing wave potential"



In this case, $C_n = 0$ for all n.

To have $C_n \neq 0$, we need a potential of the form

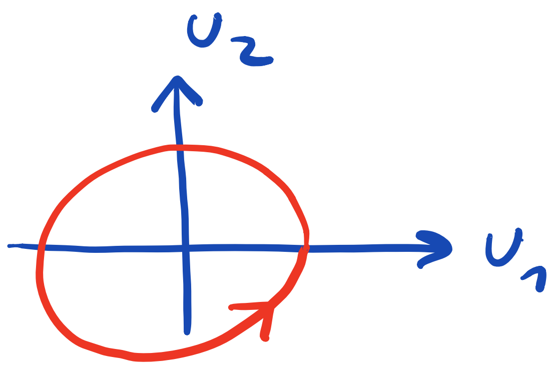
$$U(x,t) = U_1(t)\, f_1(x) + U_2(t)\, f_2(x)$$

where

$$f_1(x) = f_1(x+a)$$

$$f_2(x) = f_2(x+a)$$

but $f_1(x) \neq f_2(x)$



Need U_1 and U_2 to make a loop of nonzero area

The sliding potential can be written in such form:

$$U(x - vt) =$$

$$U(x) = U_0 \sin\left(\frac{2\pi}{a}x\right)$$

$$= U_0 \sin\left(\frac{2\pi}{a}x - \frac{2\pi}{T}t\right)$$

$$= U_0 \cos\left(\frac{2\pi}{T} t\right) \sin\left(\frac{2\pi}{a} x\right)$$

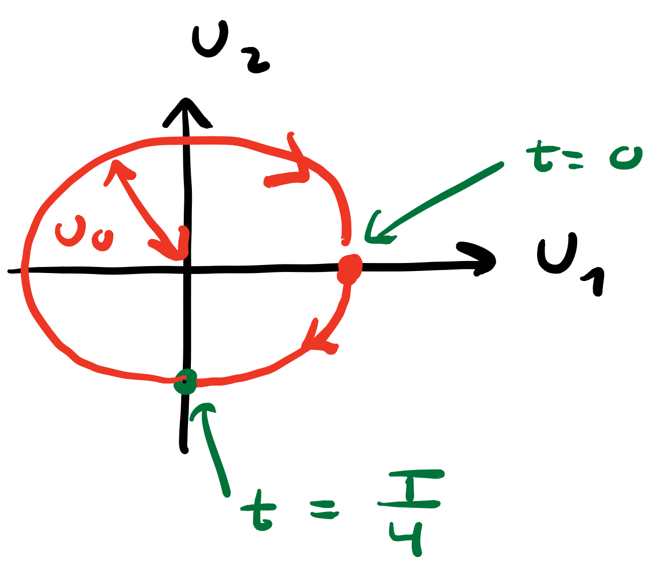
$$- U_0 \sin\left(\frac{2\pi}{T} t\right) \cos\left(\frac{2\pi}{a} x\right)$$

$$U_1(t) \equiv U_0 \cos\left(\frac{2\pi}{T} t\right)$$

$$U_2(t) \equiv -U_0 \sin\left(\frac{2\pi}{T} t\right)$$

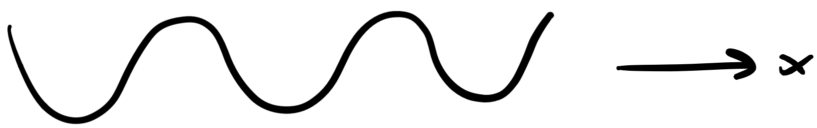
$$f_1(x) = \sin\left(\frac{2\pi}{a} x\right)$$

$$f_2(x) = \cos\left(\frac{2\pi}{a} x\right)$$



2.2 Experimental observation

Cold atoms in an optical lattice.



Nakajima et al., Nature Physics 12, 296 (2016)

Lohse et al., Nature Physics 12, 350 (2016)

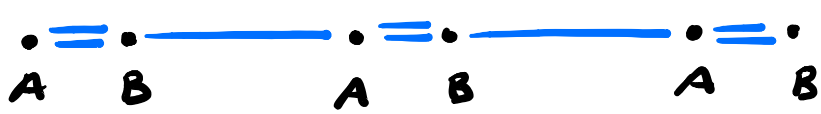
3 Su-Schrieffer-Heeger (SSH) model

Originally introduced to describe polyacetylene chains.

1D dimerized lattice.

Peierls instability in 1D

$\Rightarrow$ spontaneous dimerization



Because of dimerization,

unit cell contains two atoms

(A, B).

* Tigth - binding Hamiltonian

(spinless electrons)

Preliminaries :

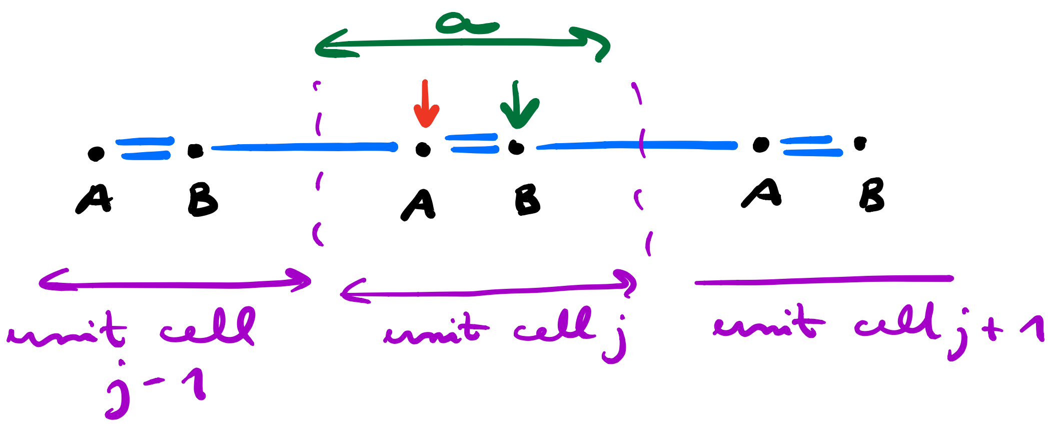
j = unit cell label

$t + \delta t$ = hopping amplitude for =

$t - \delta t$ = " " for —

c_{jA}^{+} = operator that creates an electron on unit cell j and atom A

c_{jA} = operator that destroys an electron on (j, A)



$$\mathcal{H} = \sum_j \Big[(t+\delta t)\, c_{jA}^{\dagger} c_{jB} + (t-\delta t)\, c_{jA}^{\dagger} c_{j-1\,B} + (t+\delta t)\, c_{jB}^{\dagger} c_{jA} + (t-\delta t)\, c_{jB}^{\dagger} c_{j+1\,A} \Big] \quad (1)$$

Do a Fourier transform:

$$c_{j\alpha} = \frac{1}{\sqrt{N}} \sum_k e^{ikx_j} c_{k\alpha} \quad (2)$$

↳ # of unit cells

where $\alpha = A, B$

$x_j = a j$ = position of

unit cell j.

Substitute (2) in (1), and

use the relation

$$\sum_j e^{i(k-k')x_j} = N \delta_{kk'}$$

to get

$$\mathcal{H} = \sum_k \Big[(t+\delta t)\, c^+_{kA} c_{kB} + (t-\delta t)\, e^{-ika} c^+_{kA} c_{kB} + h.c. \Big]$$

$$(t+\delta t)\, c^{\dagger}_{kA}\, c_{kB}$$

$$\xrightarrow{h.c.} (t+\delta t)\, c^{\dagger}_{kB}\, c_{kA}$$

$$(t-\delta t)\, e^{-ika}\, c^{\dagger}_{kA}\, c_{kB}$$

$$\xrightarrow{h.c.} (t-\delta t)\, e^{ika}\, c^{\dagger}_{kB}\, c_{kA}$$

$$\mathcal{H} = \sum_{k} \left(c^{\dagger}_{kA},\ c^{\dagger}_{kB} \right) \underbrace{h(k)}_{2\times 2 \text{ matrix}} \begin{pmatrix} c_{kA} \\ c_{kB} \end{pmatrix}$$

where

$$h(k) = d_x(k)\,\sigma^x + d_y(k)\,\sigma^y + d_z(k)\,\sigma^z$$

where

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

sublattice pseudospin (A, B)

$$d_x(k) = t + \delta t + (t - \delta t)\cos(ka)$$

$$d_y(k) = (t - \delta t)\sin(ka)$$

$$d_z(k) = 0$$

$$h(k)\,|u_k\rangle = E_k\,|u_k\rangle$$

$$h(k) = h\left(k + \frac{2\pi}{a}\right)$$

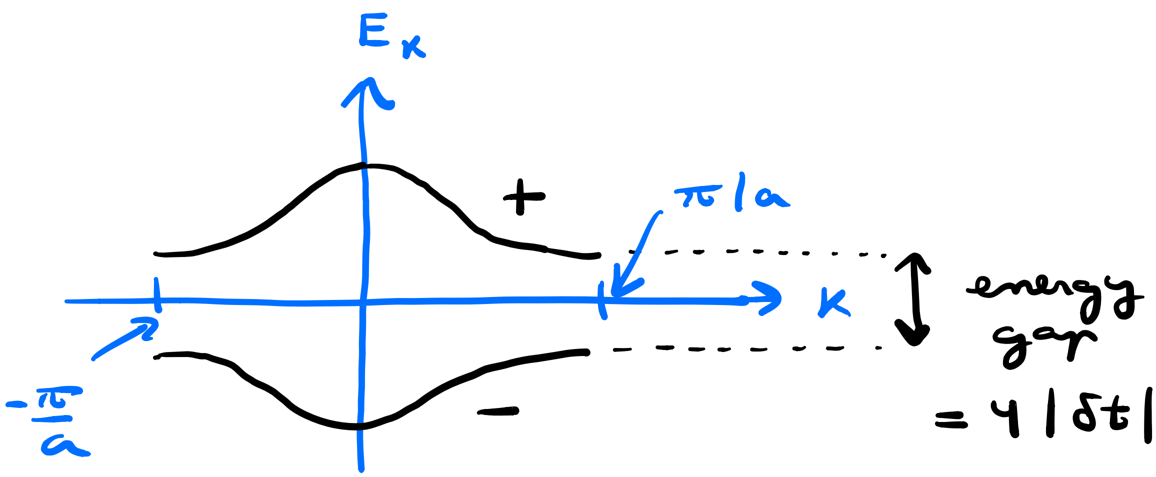
Two bands.

$$E_{k\pm} = \pm\,|\vec{d}(k)|$$

problem of a
spin in a
magnetic field.

$$= \pm \sqrt{d_x(k)^2 + d_y(k)^2 + d_z(k)^2}$$

$$= \pm \sqrt{2(t^2 + \delta t^2) + 2(t^2 - \delta t^2)\cos(ka)}$$



This corresponds to an insulator if the lower band is completely filled and the upper band is empty.

$\Leftrightarrow$ 1 electron per unit cell

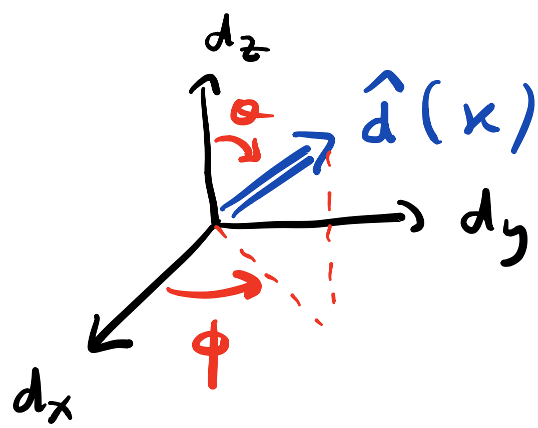
$\Leftrightarrow$ "half-filling"

Eigenstates:

$$|u_{k,+}\rangle \doteq \begin{pmatrix} \cos\left(\frac{\theta_k}{2}\right) \\ e^{i\varphi_k} \sin\left(\frac{\theta_k}{2}\right) \end{pmatrix}$$

$$|u_{k,-}\rangle \doteq \begin{pmatrix} -\sin\left(\frac{\theta_k}{2}\right) \\ e^{i\varphi_k} \cos\left(\frac{\theta_k}{2}\right) \end{pmatrix}$$

$$\vec{d}(k) = (d_x(k), d_y(k), d_z(k))$$



In the SSH model, $d_z = 0$.

$\Rightarrow \quad \theta_k = \frac{\pi}{2}$ for all k

$$\Rightarrow \quad |u_{k,+}\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix}$$

$$|u_{k,-}\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ e^{i\varphi} \end{pmatrix}$$