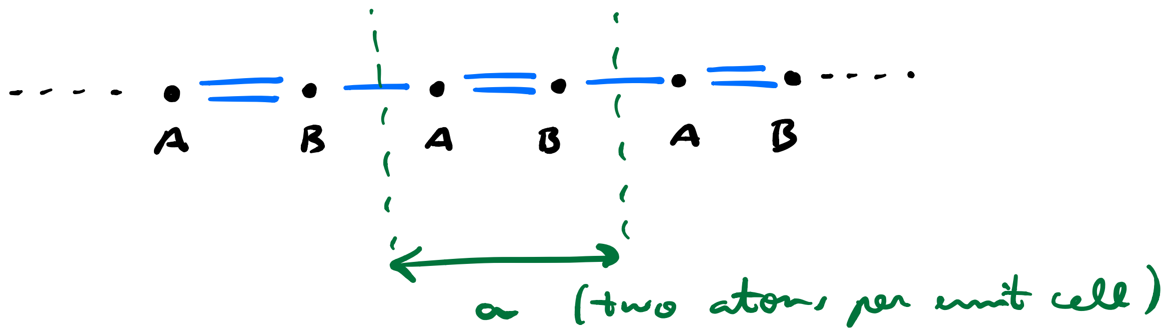
③ SSH model

spinless e^--s.



$=$ hopping amplitude $t + \delta t$

$-$ " " $t - \delta t$

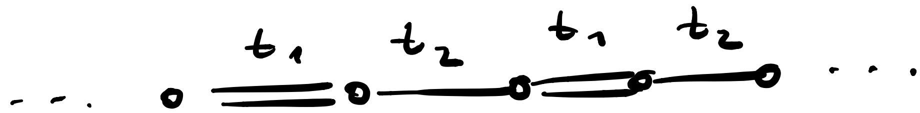
$t, \delta t \in \mathbb{R}$

$t > 0$ w/o loss of generality

$\delta t > 0$ or < 0. $\qquad t > |\delta t|$

Hamiltonian in 2nd quantized form:

$$\mathcal{H} = \sum_k \left(c^+_{k,A} , c^+_{k,B} \right) h(k) \begin{pmatrix} c_{k,A} \\ c_{k,B} \end{pmatrix}$$



$$t_1 = \overbrace{\frac{t_1 + t_2}{2}}^{t} + \overbrace{\frac{t_1 - t_2}{2}}^{\delta t}$$

$$t_2 = \underbrace{\frac{t_1 + t_2}{2}}_{t} - \underbrace{\frac{t_1 - t_2}{2}}_{\delta t}$$

where $h(k) = \vec{d}(k) \cdot \vec{\sigma}$

sublattice pseudospin

Like a spin $\frac{1}{2}$ particle in a "magnetic field" $\vec{d}(k)$.

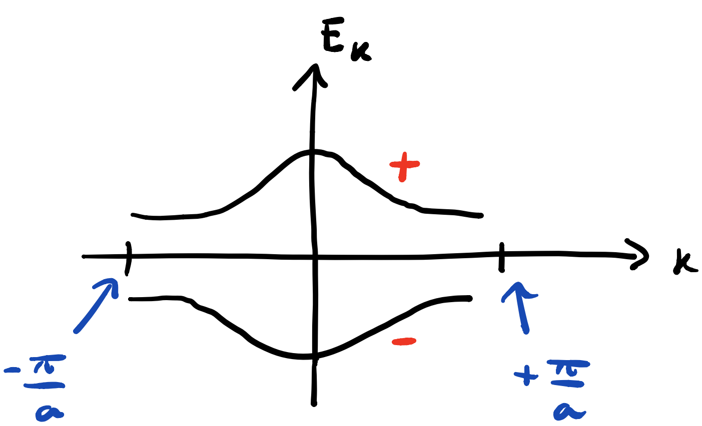
$$d_x(\vec{k}) = t + \delta t + (t - \delta t)\cos(ka)$$

$$d_y(\vec{k}) = (t - \delta t)\sin(ka)$$

$$d_z(\vec{k}) = 0$$

Periodicity $\frac{2\pi}{a}$

$$h(k)\,|u_k\rangle = E_k\,|u_k\rangle$$



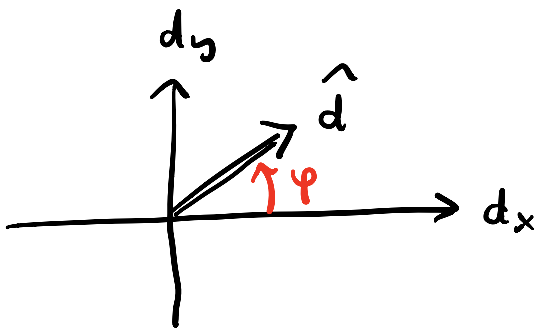
$\updownarrow$ gap $= 4|\delta t|$

the sign of δt will be important later.

Insulator at half-filling.

$$|u_{k+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix}$$

$$|u_{k-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ e^{i\varphi} \end{pmatrix}$$



$$\vec{d}(k) = \vec{d}\left(k + \frac{2\pi}{a}\right)$$

$$\Rightarrow \quad E_{k\pm} = E_{k+\frac{2\pi}{a},\pm}$$

$$|u_{k,\pm}\rangle = |u_{k+\frac{2\pi}{a},\pm}\rangle$$

(single-valued basis)

3.1 Berry phase

$$\underbrace{\gamma_{\pm}}_{} = i \int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} \langle u_{k,\pm} | \frac{d}{dk} | u_{k,\pm} \rangle \, dk$$

defined
mod 2π

$$= i \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \, \frac{1}{2} \left(\pm 1, e^{-i\varphi} \right) \begin{pmatrix} 0 \\ i e^{i\varphi} \frac{d\varphi}{dk} \end{pmatrix}$$

$$= - \frac{1}{2} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d\varphi}{dk} \, dk$$

$$= - \frac{1}{2} \left[\varphi\left(\frac{\pi}{a}\right) - \varphi\left(-\frac{\pi}{a}\right) \right]$$

$$= - \frac{1}{\cancel{2}} \cancel{2} \pi n = - \pi n$$

n = winding # = # of times

$\hat{d}$ goes around the origin

in the (d_x, d_y) plane as

k is varied from $-\frac{\pi}{a}$ to

$+\frac{\pi}{a}$.

Two possibilities:

(i) n = even

$\Rightarrow \gamma_\pm = 0 \mod 2\pi$

can be "gauged away"

"trivial"

(ii) $n = \text{odd}$

$\Rightarrow \gamma_{\pm} = \pi \mod 2\pi$

cannot be gauged away.

"nontrivial"

* Winding # for the SSH model?

Let's look at the path of $\vec{d}$ as a function of k.

(i) $k = -\frac{\pi}{a}$

$d_x = 2\,\delta t$

$d_y = 0$

(ii) $\kappa = -\dfrac{\pi}{2a}$

$$d_x = t + \delta t$$

$$d_y = -(t - \delta t)$$

(iii) $\kappa = 0$

$$d_x = 2t$$

$$d_y = 0$$

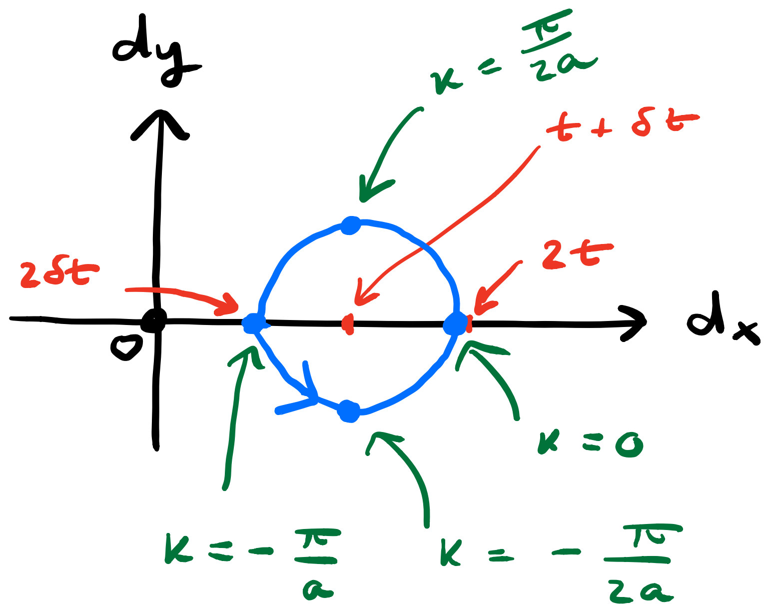
(iv) $\kappa = \dfrac{\pi}{2a}$

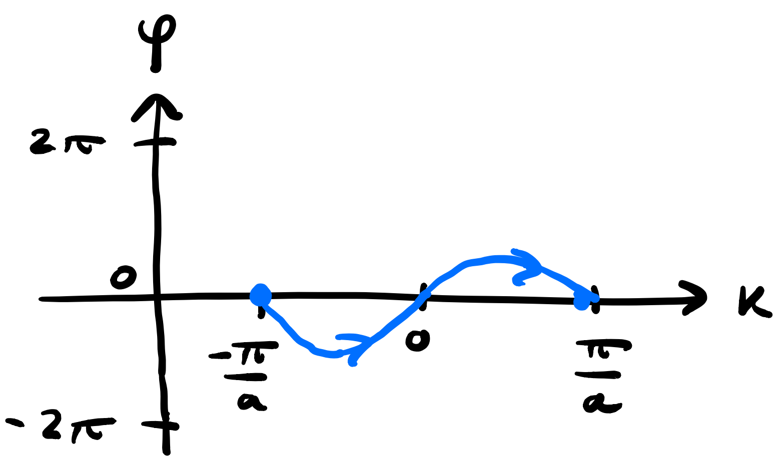
$$d_x = t + \delta t$$

$$d_y = t - \delta t$$

Assume (w/o loss of generality) that $t > |\delta t|$

$$\boxed{\text{Case \# 1 : } \delta t > 0}$$

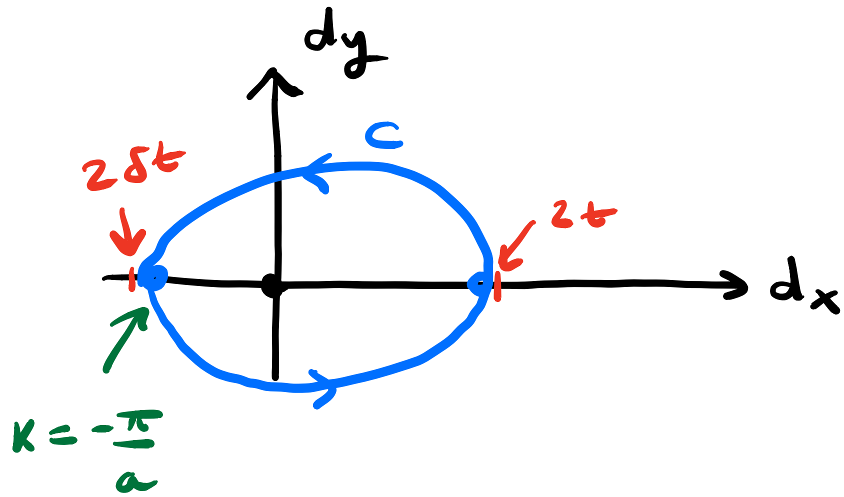


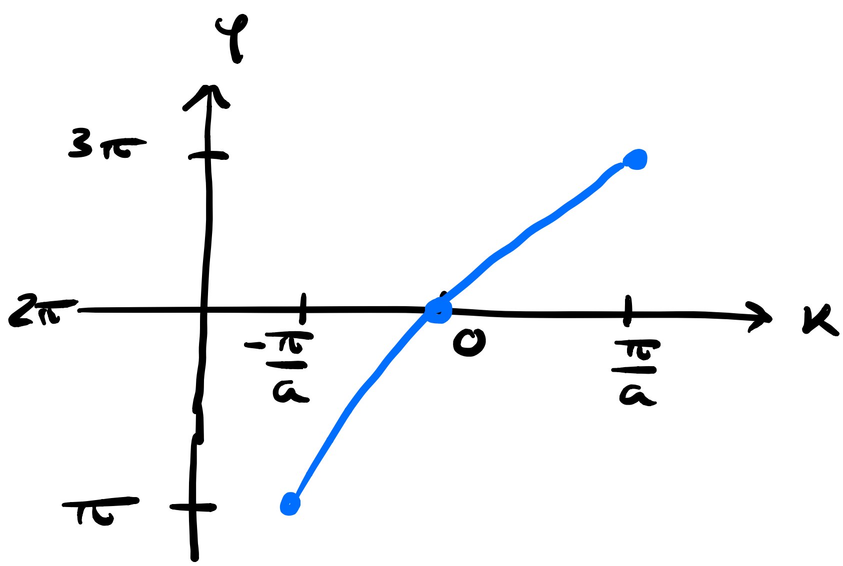


$$\oint_C d\varphi = 0$$

$$\boxed{n = 0}$$

Case # 2 : $\delta t < 0$





$$\oint_c d\varphi = 2\pi$$

$$\boxed{n = 1}$$

* Alternative viewpoint :

$$h(\vec{d}) = \vec{d} \cdot \vec{\sigma}$$

$\vec{d}$ makes a closed loop C

as k is varied from $-\frac{\pi}{a}$

to $+\frac{\pi}{a}$.

"spin" 1/2

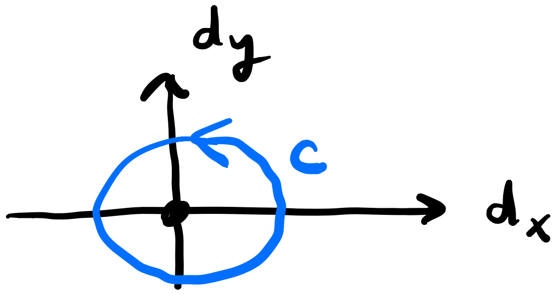
$$\gamma_\pm = \pm \frac{1}{2} \Omega$$

lecture 4

solid angle
subtended by C
on the unit
sphere.

$$\Omega = \int d\theta \sin\theta \int d\varphi$$

Case $\delta t < 0$:



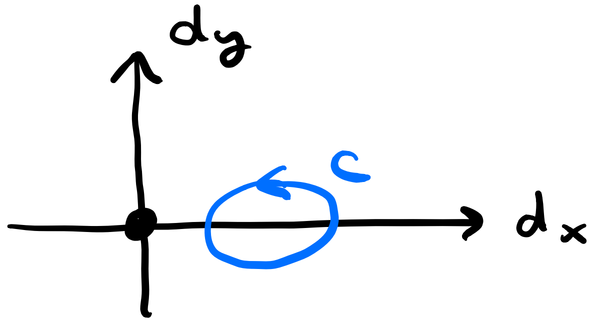
$$\Omega = \underbrace{\int_0^{\pi/2} d\theta \sin\theta}_{} \underbrace{\int d\varphi}_{}$$

$$= 1 \times 2\pi$$

$$\Rightarrow \gamma_\pm = \pm \pi$$

$$(\gamma_\pm = \pi)$$

Case $\delta t > 0$:



$$\int d\varphi = 0 \Rightarrow \Omega = 0$$

$$\Rightarrow \gamma_{\pm} = 0$$

Note also that the Berry curvature,

$$\vec{B}_{\pm} = \pm \frac{\hat{d}}{2d^2} ,$$

has a monopole at $\vec{d} = 0$.

We get a nontrivial Berry phase if the path C encloses the monopole.

3.2 Topological phase diagram

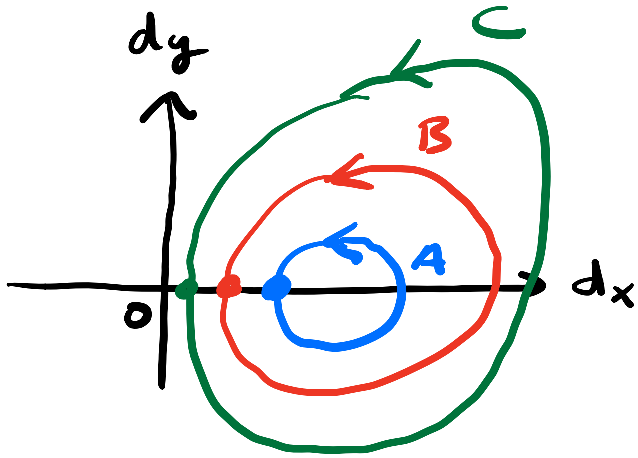
Consider perturbations to the SSH Hamiltonian, under two conditions:

(i) keep $d_z(k) = 0$, $\forall k$.

(ii) do not close the gap ($\delta t \neq 0$)

What happens to the winding # n? Nothing.

$\rightarrow$ n = "topological invariant"

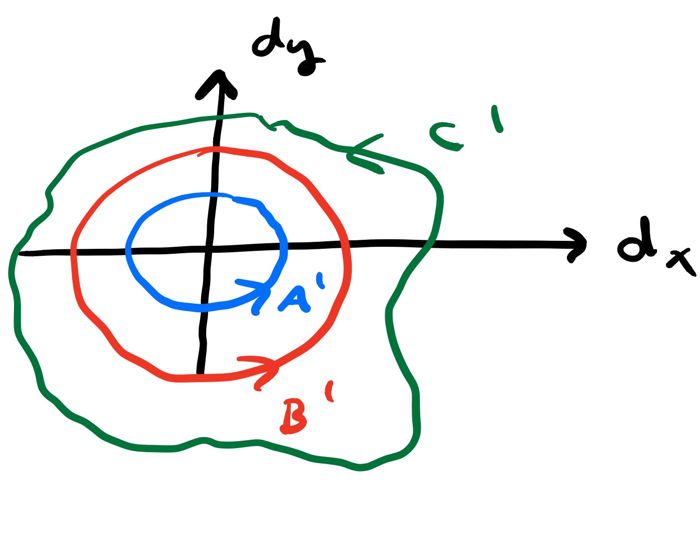


A, B, C correspond to

3 different Hamiltonians

w/ $d_z = 0$ and $\delta t > 0$

They all have $\gamma_{\pm} = 0$



A', B', C' correspond to 3 different $h(k)$ w/ $d_z = 0$ and $\delta t < 0$.

They all have $\gamma_{\pm} = \pi$

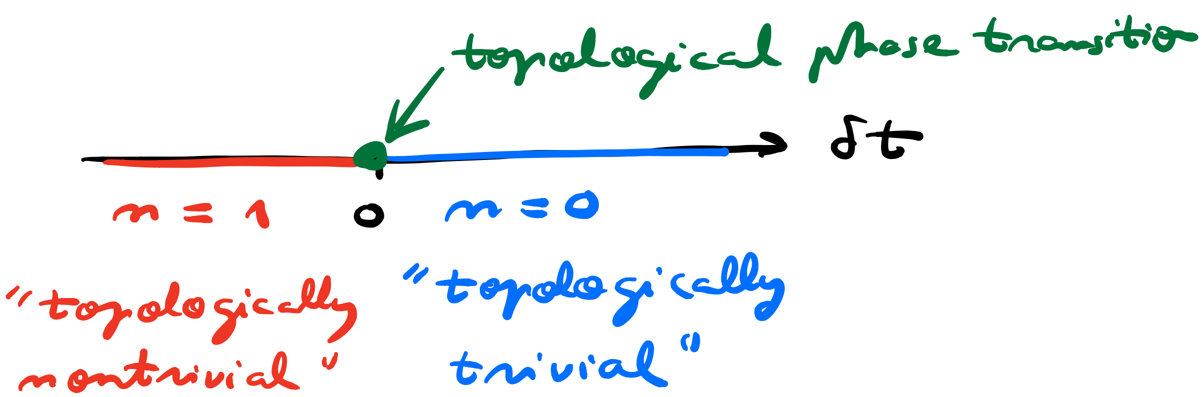
* "Topological classification":

$$\left.\begin{array}{l}\left.\begin{array}{l}A\\ B\\ C\end{array}\right\}\text{topologically equivalent}\\ \\ \left.\begin{array}{l}A'\\ B'\\ C'\end{array}\right\}\text{topologically equivalent}\end{array}\right\}\text{topologically distinct}$$

"topologically equivalent": they can be related to one another by smooth deformations of $h(k)$.

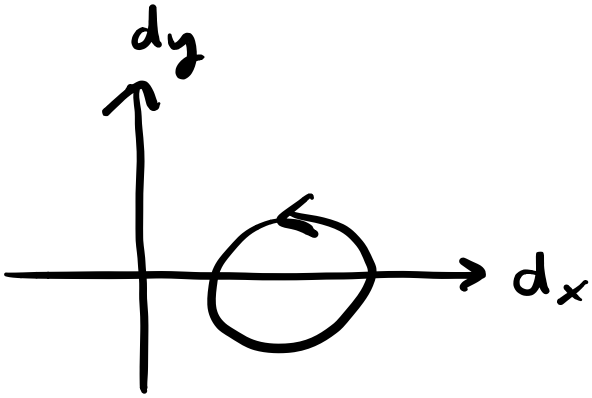
"smooth deformation":
deformation that does not close the energy gap of the insulator.

"Topological phase diagram:"

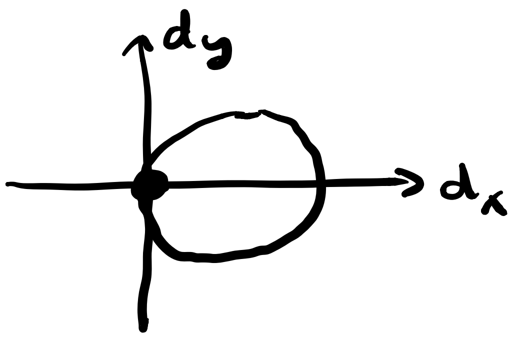


Energy gap closes at transition.

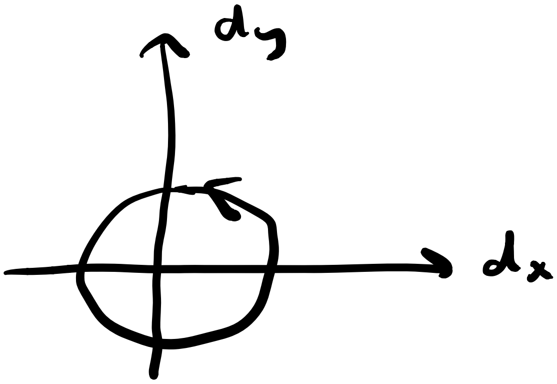
To go from one phase to another, we need to close the gap:



change δh $\Longrightarrow$

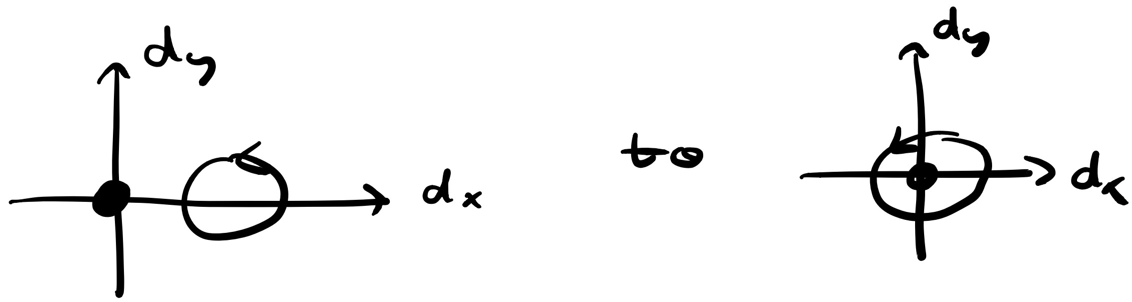


change δh $\Longrightarrow$



* The topological classification no longer applies if we allow $d_z \neq 0$.

If $d_z \neq 0$, we can go from



w/ smooth deformations (i.e. w/o closing the gap)

Thus, A and A' (for example) are no longer topologically distinct when $d_z \neq 0$.

Likewise, when $d_z \neq 0$,

$r_\pm$ can be anything between

0 and π.

* Physical meaning of $d_z = 0$:

Symmetry.

$$h(k) = d_x \sigma^x + d_y \sigma^y + d_z \sigma^z$$

If $d_z = 0$, $\boxed{\{h(k), \sigma^z\} = 0}$

$$\{A, B\} = AB + BA$$

$$\{\sigma^i, \sigma^j\} = 2\delta_{ij}$$

"chiral symmetry"

Consider

$$h(k)\,|u_k\rangle = E_k\,|u_k\rangle$$

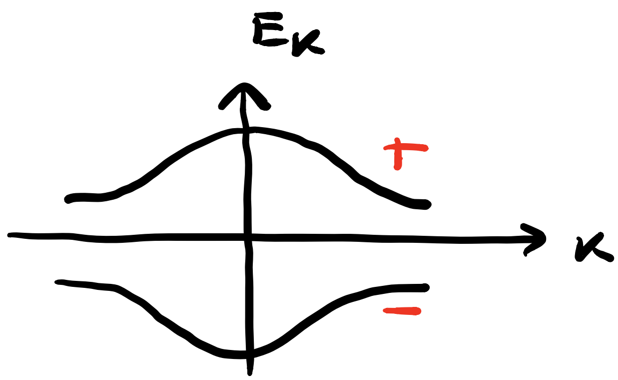
$$\Rightarrow \sigma^z h(k)\,|u_k\rangle = E_k\,\sigma^z\,|u_k\rangle$$

$$\Rightarrow -h(k)\,\sigma^z\,|u_k\rangle = \quad ''$$

chiral symmetry

$$\Rightarrow h(k)\left(\sigma^z\,|u_k\rangle\right) = -E_k\left(\sigma^z|u_k\rangle\right)$$

Message: if $|u_k\rangle$ is an eigenstate of energy E_k, then $\sigma^z\,|u_k\rangle$ is an eigenstate of energy $-E_k$.



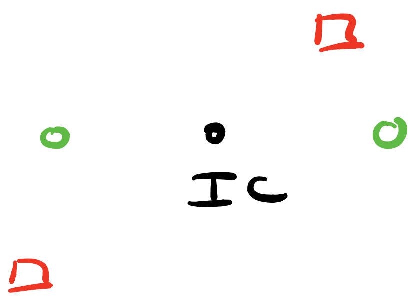
$$E_{k-} = - E_{k+}$$

The Hamiltonians giving rise to A and A' are topologically distinct only if the system has a chiral symmetry.

" $\underbrace{S}_{\text{symmetry}}\underbrace{P}_{\text{protected}}\underbrace{T}_{\text{topological}}$ " phases

In 1D, if all symmetries are broken, then all insulators are topologically equivalent (all "trivial")

The same is not true in 2D



$|u_k\rangle \qquad \sigma^z |u_k\rangle$

$$|u_{k+}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix}$$

$$|u_{k-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ e^{i\varphi} \end{pmatrix}$$

$$\underline{\underline{\sigma^z |u_{k+}\rangle}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |u_{k+}\rangle$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{i\varphi} \end{pmatrix}$$

$$= e^{i\pi} \underline{\underline{|u_{k-}\rangle}}$$

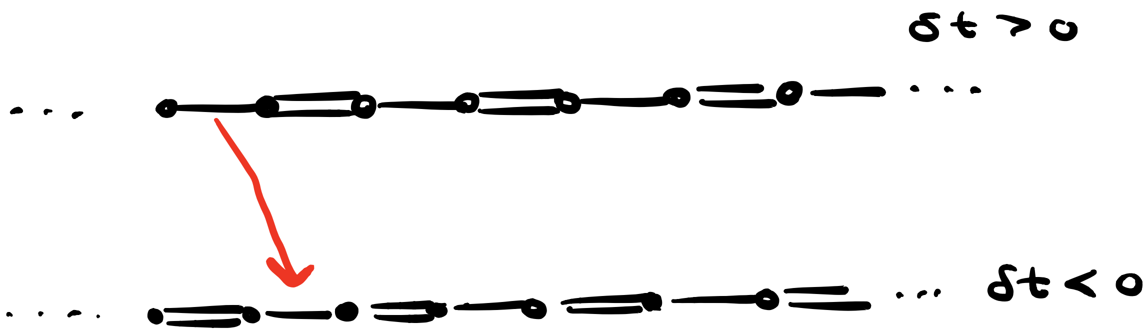
3.3 Zero-energy edge modes

Chiral symmetry

$\rightarrow$ $\delta t > 0$ and $\delta t < 0$ are

topologically distinct.

Is this difference observable?



For an ∞ chain, or for a ring, there's no way to distinguish between the two phases b/c they are related by a lattice translation.

The distinction between $\delta t > 0$ and $\delta t < 0$ arises physically in

(i) finite-sized chain (hw #3)

(ii) domain walls (in class)

A domain wall: