

**COMMENT ON THE ROLE OF THERMODYNAMIC  
REPRESENTATIONS IN THE STUDY OF FLUIDS IN  
FAR FROM EQUILIBRIUM STEADY STATES**

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It is shown that calculations of fluctuations in fluids driven into a stationary state by a temperature gradient are independent of the thermodynamic representation even to nonlinear order in the temperature gradient. The contrast between this result and the conjecture of Garibay-Jiménez and García-Colin, *Physica* **130A** (1985) 616, is clarified.

The problem of calculating fluctuations in systems not in thermal equilibrium is a long standing one<sup>1</sup>). The special case of hydrodynamic fluctuations about simple dissipative stationary states has come to the forefront in the last six years or so. Various approaches<sup>2</sup>) all lead to the same result in the test case of Brillouin scattering in a simple fluid subjected to a temperature gradient. Experiment also confirms quantitatively the theory<sup>3</sup>). Early calculations were done to linear order in the applied external gradient. Later, effects nonlinear in the applied external stresses were studied<sup>4-7</sup>).

Recently, Garibay-Jiménez and García-Colin<sup>8</sup>) (referred to as GG) have extended previous calculations<sup>6-7</sup>) of the Rayleigh peak which had been performed to second order in the applied temperature gradient. The main point of their paper is that to second order only one thermodynamic representation seems to be adequate for such a calculation. In this brief note, we would like to support the opposite point of view.

There are two types of hypotheses behind the calculation of fluctuations about nonequilibrium steady states from the fluctuating hydrodynamics point of view: dynamical and statistical hypotheses. In the dynamic part, one assumes that the

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behavior of the hydrodynamic variables can be computed from the deterministic hydrodynamic equations: first, one solves the full nonlinear hydrodynamic equations for the steady states; second, one linearizes the full equations around the steady state. On the statistical side, one assumes that the Langevin noise terms should be added to the linearized equations and that they take their local equilibrium form. These hypotheses have been discussed at length<sup>9-10</sup>) and are generally accepted for the kind of problems we discuss here. The questions raised by GG are at the dynamical level hence we will have nothing more to say about Langevin forces.

Consider the nonlinear hydrodynamic equations<sup>11</sup>):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \Pi, \quad (2)$$

$$\rho T \left( \frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s \right) = -\nabla \cdot \mathbf{J}_Q. \quad (3)$$

Here  $\rho$  is the mass density,  $\mathbf{v}$  the fluid velocity,  $p$  the pressure,  $T$  the temperature,  $s$  the entropy per unit mass, while  $\Pi$  and  $\mathbf{J}_Q$  are the *dissipative parts* of, respectively, the stress tensor and the heat flux. For the purpose of our discussion, these dissipative fluxes need not be written explicitly.

When two sides of the system are maintained at a constant temperature difference  $\Delta T$ , the steady state solution of eqs. (1) to (3) is,  $\mathbf{v} = \mathbf{0}$ ,  $p = \text{constant}$  and  $\nabla T = \text{constant}$ . Note that the temperature gradient is a constant only to first order in  $\Delta T/T$  or more generally when the temperature dependence of the heat conductivity can be neglected.

In the next step, the nonlinear hydrodynamic equations are linearized around the steady state:

$$\frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho \delta \mathbf{v}) = 0, \quad (4)$$

$$\rho \frac{\partial \delta \mathbf{v}}{\partial t} = -\nabla \delta p - \nabla \cdot (\delta \Pi), \quad (5)$$

$$\rho T \left( \frac{\partial \delta s}{\partial t} + \delta \mathbf{v} \cdot \nabla s \right) = -\nabla \cdot (\delta \mathbf{J}_Q). \quad (6)$$

The fluctuations, which are driven by Langevin noise terms added to the dissipative fluxes, are denoted by  $\delta$ . It is very important to keep in mind that *all* nonfluctuating quantities appearing in eqs. (4) to (6) have their local steady state value, i.e. in general they depend on position because the temperature depends

on position in the steady state. Eqs. (4) to (6) are linearized in the fluctuations but not in  $\nabla T$ . Eqs. (4) to (6) correspond for example to eqs. (2.9)–(2.11) of van der Zwan et al.<sup>13</sup>).

The usual assumptions of hydrodynamics now allow us to express the fluctuating thermodynamic quantities in terms of only two independent variables. When changing representation, it is important to keep in mind that the thermodynamic derivatives appearing in equations such as

$$\delta p = \left( \frac{\partial p}{\partial \rho} \right)_s \delta \rho + \left( \frac{\partial p}{\partial s} \right)_\rho \delta s \quad (7)$$

do depend on position so that in computing  $\nabla \delta p$  for example, the gradient acts on the thermodynamic derivatives as well<sup>12</sup>). Note that similar care must be taken when relating the Fourier transforms of fluctuating quantities in different representations.

In the  $(\rho, T, \mathbf{v})$  representation, one obtains

$$K_T \frac{\partial \delta p}{\partial t} - \beta \frac{\partial \delta T}{\partial t} = \beta \delta \mathbf{v} \cdot \nabla T - \nabla \cdot \delta \mathbf{v}, \quad (8)$$

$$\rho \frac{\partial \delta \mathbf{v}}{\partial t} = -\nabla \delta p - \nabla \cdot (\delta \Pi), \quad (9)$$

$$\frac{\partial \delta T}{\partial t} - \beta \frac{T}{c_p} \frac{\partial \delta p}{\partial t} = -\delta \mathbf{v} \cdot \nabla T - \frac{1}{c_p} \nabla \cdot (\delta J_Q), \quad (10)$$

which are the equations used by GG<sup>8</sup>) and by Ronis and Procaccia<sup>7</sup>). Here  $K_T = \rho^{-1}(\partial \rho / \partial p)_T$ ,  $\beta = -\rho(\partial s / \partial p)_T$  and  $c_p = \rho T(\partial s / \partial T)_p$ . In the  $(\rho, s, \mathbf{v})$  representation, the corresponding equations derived from eqs. (4)–(6) are

$$\frac{\partial \delta p}{\partial t} - \rho \beta \delta \mathbf{v} \cdot \nabla T + \rho \nabla \cdot \delta \mathbf{v} = 0, \quad (11)$$

$$\rho \frac{\partial \delta \mathbf{v}}{\partial t} = -\nabla \left[ \left( \frac{\partial p}{\partial \rho} \right)_s \delta \rho + \left( \frac{\partial p}{\partial s} \right)_\rho \delta s \right] - \nabla \cdot \delta \Pi, \quad (12)$$

$$\rho T \left( \frac{\partial \delta s}{\partial t} + \frac{c_p}{\rho T} \delta \mathbf{v} \cdot \nabla T \right) = -\nabla \cdot (\delta J_Q). \quad (13)$$

When steady state quantities are expanded around an average state to linear order in  $\nabla T$ , the above equations lead immediately to eqs. (4.3)–(4.5) of van der Zwan et al.<sup>13</sup>). Using a few standard thermodynamic identities, the Reader can convince himself that it is possible to go from Eqs. (8)–(10) to eqs. (11)–(13) and vice-versa. GG have stated that while it is possible to go from (8)–(10) to (11)–(13), the reverse transformation is not possible. They also

remark on  $\delta\mathbf{v} \cdot \nabla T$  terms\* missing in the  $(\rho, s, \mathbf{v})$  representation. These discrepancies between the analysis of GG and our work arise when GG assume that a choice of  $\rho, s, \mathbf{v}$  as independent variables means that one can take  $\nabla s = \nabla \rho = \mathbf{0}$  in the stationary state even if there is an applied temperature gradient. Our understanding of fluctuating hydrodynamics is that all non-fluctuating quantities appearing in the linearized hydrodynamic equations (4)–(6) must take their local value and thus depend in general on position when the temperature does. In this case, as we have shown above, the calculated fluctuations are independent of which thermodynamic representation one is using, even to non-linear order in the applied gradient.

Finally, we should like to recall that it is not inconsistent to expand the steady-state variables about a uniform average state to first order in  $\nabla T$  and to use the resulting equations to compute the fluctuations to second order in  $\nabla T$ . The expansion parameters involved in both expansions are in general different. In the case of the fluctuations, decay lengths of hydrodynamic modes are also involved in the expansion parameter. In either thermodynamic representation, the fluctuations for the case of interest have to be computed to second order in  $\nabla T$  and then properly multiplied and averaged over the Langevin forces to compute the structure factor. Such a procedure was used before, for example, in ref. 7 and also in ref. 5 for a different but analogous nonlinear problem. One thermodynamic representation may be easier to manipulate than the other but the results have to be equivalent.

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### References

- 1) See M. Lax, *Rev. Mod. Phys.* **32** (1960) 25; and references therein.
- 2) See A.-M.S. Tremblay, in: *Recent Developments in Nonequilibrium Thermodynamics, Lecture Notes in Physics*, vol. 199, J. Casas-Vázquez, D. Jou and G. Lebon, eds. (Springer, Berlin, 1984); and references therein.

\*The  $\delta\mathbf{v} \cdot \nabla T$  discussed by GG does not come from the dissipative parts. That is why we have not explicitated them.

- 3) H. Kiefte, M.J. Clouter and R. Penney Phys. Rev. B **30** (1984) 4017.  
D. Beysens, Y. Garrabos, and G. Zalczer, Phys. Rev. Lett. **45** (1980) 403.
- 4) T.R. Kirkpatrick and E.G.D. Cohen, Phys. Lett. **78A** (1980) 350.  
H. Ueyama, J. Phys. Soc. Japan **51** (1982) 3343.
- 5) C. Tremblay and A.-M.S. Tremblay, Phys. Rev. A **25** (1982) 1692.
- 6) T.R. Kirkpatrick, E.G.D. Cohen and J.R. Dorfman, Phys. Rev. A **26** (1982) 995.
- 7) D. Ronis and I. Procaccia, Phys. Rev. A **26** (1982) 1812.
- 8) R. Garibay-Jiménez and L.S. García-Colin, Physica **130A** (1985) 616. (Referred to as GG in the text.)
- 9) See, for example, J. Keizer, Phys. Fluids **21** (1978) 198.  
D. Ronis, I. Procaccia, and J. Machta, Phys. Rev. A **22** (1980) 714.  
T.R. Kirkpatrick, E.G.D. Cohen and J.R. Dorfman, Phys. Rev. A **26** (1982) 950.
- 10) See A.-M.S. Tremblay, M. Arai, and E.D. Siggia, Phys. Rev. A **23** (1981) 1451.
- 11) L.D. Landau and E.M. Lifshitz, Fluid Mechanics (Pergamon, London, New York) eqs. (1.2), (15.6), (49.5) and sec. 132.
- 12) See ref. 10, p. 1463 for more details.
- 13) G. van der Zwan, D. Bedeaux and P. Mazur, Physica **107A** (1981) 491.