ELECTRICAL PROPERTIES OF FRAC TAL NETWORKS (AND WHY)

A. H. S. Tremblay

Laboratory for Atomic and Solid State Physics
and Materials Science Center
Clark Hall, Cornell University, Ithaca, NY 14853-2901

ABSTRACT

Close to the critical concentration at which a mixture of metal and insulator changes from insulating to conducting, the electrical properties of the mixture are a power law function of a macroscopic length, the correlation length. Qualitative insight into the origin of these power laws can be gained by considering exactly self-similar structures. The mass of these structures, for example, is a power law of the length scale (the fractal dimension). Using mainly the hierarchical lattice of De Arcangelis et al. as an illustration, it is shown that the fractal dimension does not suffice to describe either the resistance or the resistance noise of self-similar systems. Certain frequency dependent properties however do not involve new exponents. It is shown that on the other hand a different exponent is needed to describe each different cumulant of the resistance fluctuations. These measurable exponents suffice to characterize completely the current distribution. The asymptotic form of this distribution is also discussed. The examples treated are the simplest ones and should provide a tutorial introduction to the subject.

INTRODUCTION

Dr. Voss, at this conference, has given many examples of naturally occurring fractal objects. Perhaps the most relevant fractal objects for this audience are percolation clusters. More details on percolation are given in the paper by Dr. Bug." In percolation, there is an intermediate concentration \( p_c \) below which a mixture of metal and insulator, for example, is insulating, and above which it is conducting. For values of \( p \) far from \( p_c \), one can compute the electrical properties of the mixture from relatively simple effective medium (or self-consistent) types of theories.(2) Close to \( p_c \) the problem is more difficult: fluctuations become enormous and physical properties vary as a power law of \( \Delta p \propto p-p_c \).

The problem becomes analogous to a second order phase transition. The origin of the difficulty can be traced to the fractal geometry of the percolating structure at length scales smaller than a length \( \xi \), the correlation length, which diverges as one approaches the percolation threshold as \( \Delta p \propto (\Delta p-p_c)^{-\nu} \). If one can compute the fractal dimension of the percolating clusters, or any other exponent characterising their physical properties, the problem becomes essentially solved: for example, in the phenomenology of percolation the probability that a site belong to the infinite cluster scales as \( \Delta p^\beta \propto \xi^{-\rho/\nu} \). This can be related to the fractal dimension by noting that within a correlation length, the number of connected sites scales as \( \xi^{d} \) which must be multiplied by the number of regions of size \( \xi \) in a system of size \( L \), i.e. \( \xi^D \) (at length scales larger than \( \xi \), the geometry of the system is Euclidean).(11) Hence, \( \beta/\nu = d/D \).

Surprisingly, objects such as the one illustrated in Fig. 1 have something to say about percolation.(4) Fig. 1 displays a 'fractal' object but a simple one: it is exactly self-similar. The percolating clusters or the fractal mountains of Dr. Voss are only statistically self-similar, i.e. it is only "on average" that they are self-similar. One of the major contributions of Dr. Hambrook to this field has in fact been to point out that exactly self-similar objects, such as the Sierpinski gasket of Fig. 1, which had been studied by mathematicians, share some qualitative properties with the statistically self-similar objects that one encounters in nature: \( \xi \approx L \) all physical properties of these two kinds of systems are described by power laws (Power law behavior and self-similarity are in a sense synonymous. Power law dependence on length scale is also known as "scaling behavior") and second, relations between exponents which apply for exactly self-similar objects, are valid also in general for statistically self-similar objects. In fact, both statistically self-similar objects and exactly self-similar objects, which one could have a priori considered very different, are today called fractals. Fractals are thus, in a sense, the intuition and the geometry behind the renormalization group, which is itself the analytical tool which one uses to compute physical properties of self-similar objects. Note that a Euclidean line or plane are also exactly self-similar objects. Their physical properties depend on length scale also as a power law but in a trivial way.

The purpose of this talk is to give a few examples of the qualitative insights on the physics of percolating systems that have been obtained through the use of fractals. (See also Ref.[9] for an analogous survey.) It should be stressed that one cannot obtain quantitative information on disordered systems by computing the properties of exactly self-similar fractals, except in a few rare instances where approximate renormalization group calculations for real disordered systems happen to be mathematically identical to the solution for some exactly self-similar fractal structure.(6)
THE RESISTANCE EXPONENT AND THE SPECTRAL "DIMENSION"

Consider the fractals of Figs. 1 or 2. We know that for percolation, the size $\xi$ of the percolating clusters increases as one gets closer to the threshold $p_c$. One can make a geometrical picture of this by imagining that Fig. 1c or 2c represent percolating clusters when $\xi$ has, respectively, two or three times the linear size of Fig. 1b or 2b. Clearly the number of bonds $N$ depends on size as

$$N \sim \xi^D$$

$D$ is in 3 / 2 for Fig. 1, and in 4 / 3 for Fig. 2. [17] As a first example of a qualitative question, let us ask how does the electrical resistance of such an object depend on $\xi$? Is the knowledge of $D$ sufficient to answer this question? We know that the answer to this question is yes in the case of Euclidean systems. For an n-polyhedron of side $L$ in $d$ dimensions, the resistance scales with $L$ as $L^d$.

Let us answer this question for the fractal of Fig. 1 since it is the simplest. This fractal is often called the "hierarchical" model of percolating systems and it has been studied in detail by a group in Boston[8]. First note that the fractal of Fig. 1c is constructed by replacing each bond in Fig. 1b by the pattern of Fig. 1c itself. In other words, the rule that leads from Fig. 1a to Fig. 1b must be repeated recursively in $\xi$ / $L$ in 3 times the size $\xi$ is given, and we know that we have a system of size $\xi$, one way to compute the resistance between the terminals on the extreme left and right is to work backwards. The resistance of the pattern -0- in Fig. 1c is $r' = (3/2)r$ if $r$ is the resistance of an elementary bond, so the resistance of Fig. 1c is identical to that of Fig. 2b if the elementary bonds of the latter have a resistance $r'$. This is an elementary example of a renormalization group: The total resistance $R_{EC}$ of a lattice of size $\xi$ such as that of Fig. 1c, with elementary resistances $r$, is equal to $1 / 2$ of the overall resistance $R_{EC} / 2\xi$ of a hierarchical lattice of size $\xi / 3$, such as that of Fig. 2b, when it is made of elementary resistances $r$.

$$R_{EC} = \frac{1}{2} R_{EC} / \xi$$

One solution to this equation is $R_{EC} = \xi^{\xi / \nu}$ with $\xi / \nu = 3/2 - 1 / 3$. By the way, the fractal dimension can be computed by a renormalization procedure entirely analogous to the one used here). Is there a relation between $\xi / \nu$ and $D$? Clearly the Euclidean relation $\xi / \nu = 2 - D$ does not hold here and the present counterexample is sufficient to invalidate it in general. But in

$$\xi / \nu = \ln(3/2) / \ln(3)$$

is generally in 3 and it is easy to invalidate the last result by considering another fractal such as the Sierpinski gasket. To show rigorously that there is no relation between $\xi / \nu$ and $D$ in general is a little bit more subtle. We come back to this question later.

As an illustration of relations between exponents which hold on both exactly self-similar and on statistically self-similar objects, let us briefly consider the AC properties of fractal networks. Let us assume that every vertex, or node, of the fractal networks we consider are connected to ground through a capacitor, and let us ask what is the frequency $\omega$-dependent admittance $Y$ measured at, for example, one of the external corners of the Sierpinski gasket. One finds[9] that in the mid-frequency range the loss angle $\delta$, defined by $\tan \delta = Y / Re Y$, has a "plateau" region where

$$\delta = \frac{\pi}{2} \left[ 1 - \frac{3}{2} \right]$$

$\delta$, the so-called "spectral" or "fractal" dimension [10] is equal to $2D/\ln(\xi / \nu)$. The relation between $\delta$, $D$, and $\xi / \nu$ has been derived for very general grounds[10] but it seems that the case of the Sierpinski gasket has been a great intuitive help to the authors. In the case of Eq. (3), it was first discovered on the Sierpinski gasket[9]. It could have been arrived at without the help of exactly self-similar fractals, but it is clear that these have played an important role. Eq. (3) has now been applied to AC electrical properties [11,12] near the percolation threshold. The problem of AC conductivity of fractal systems is treated at this conference by Dr. Liu.

THE "NOISE" EXPONENT

Another interesting electrical property is the noise. Thermal noise is governed by the resistance of the system and there is nothing new: it is known. Excess 1/f noise on the other hand is independent of the resistance of the sample. While the physical origin of 1/f noise is still uncertain[13] there are two experimentally well established results which we will use [14]: 1) 1/f noise is caused by resistance fluctuations, i.e., it is observed as voltage fluctuations $\delta V(t)$ which are proportional to the constant applied current $I$. 2) Voltage fluctuations between two segments of the same material are uncorrelated over distances larger than a few microns.

We then ask the following question: How does the magnitude of 1/f noise diverge near the percolation threshold, and is this divergence governed by a new exponent or by an exponent related to previously defined ones? To answer

![Figure 1: The Sierpinski gasket.](a)
![Figure 2: The hierarchical lattice.](c)
these questions, we first investigate them on fractal networks, for reasons which should by now be clear. To be consistent with properties (i) and (ii) of the previous paragraph, the following model is adopted: Assume that each resistance fluctuates in time around an average value $r$ independently from the other resistances. In other words, if $\alpha$ and $\beta$ are two of the resistances of the system,

$$r_\alpha = r + \delta r_\alpha$$  \hspace{1cm} (4a)

$$\langle \delta r_\alpha \rangle = 0$$  \hspace{1cm} (4b)

$$\langle \delta r_\alpha \delta r_\beta \rangle = \delta(\alpha, \beta) \delta(r_\alpha, r_\beta)$$  \hspace{1cm} (4c)

where the brackets stand for ensemble average (or in practice time average). The last $\delta$ symbol is Kronecker's, i.e., it is equal to 1 when $\alpha$ and $\beta$ are identical and 0 otherwise. Note that we assume that each of the elementary fluctuating resistances has an identical spectrum $\delta^2(r_\alpha)$, but the exact frequency dependence of this spectrum is not important for the following. We have used a $a$ for the Fourier transform variable conjugate to the time $t$.

To compute the magnitude of the resistance noise, we proceed as follows. First recall that the total resistance may be calculated from

$$R^2 = \sum r_i^2$$  \hspace{1cm} (5)

where $i$ is the total resistance current and $i$ is the current in branch $a$. To compute the overall resistance fluctuations of the circuit, we use well-known theorems in network sensitivity analysis. Cohn's theorem, which is a direct consequence of Tolle's theorem, shows that to linear order in the fluctuations

$$\delta R^2 = \sum \delta r_i^2$$  \hspace{1cm} (6)

Normalizing the input current to unity, and using the model of Eqs. (4), one obtains

$$\langle \delta E_\alpha \delta E_\beta \rangle = \delta(\alpha, \beta) \delta(r_\alpha, r_\beta)$$  \hspace{1cm} (7)

To find the size dependence of Eq. (7) for the fractal of Fig. 2a, assume that we know the result for Fig. 2b. Let us call this result $\delta R^2(2)/3$ for short. To find $\delta R^2(2)$ in Fig. 2c, first recall that the input current in Fig. 2b was unity. Then $\delta R^2(2)$ in Fig. 2c, first recall that the input current in Fig. 2b was unity. Then it suffices to multiply $\delta R^2(2)/3$ by the fourth power of the total current flowing through the skeleton AB, BD, and CD in Fig. 2c and to add up the results to obtain $\delta R^2(2)$ (The total current flowing from A to D is taken as unity.) The result is then

$$\delta R^2(2) = 2 \left[ 1 + \frac{1}{2} \right] \delta R^2(2)/3$$  \hspace{1cm} (8)

One solution is,

$$\delta R^2(2) \propto \ln(17/8)$$  \hspace{1cm} (9)

The exponent is clearly different from the resistance exponent of the fractal dimension and there is no simple relation linking them. Eq. (9) shows that the Euclidean result $\delta R^2(2) \propto L^{-3\phi/4}$ does not generalize to fractal structures. It suffices to consider a few more fractal structures to realize that there is generally no relation between the exponent for the noise and previously defined exponents.

As for the resistance exponent, Monte Carlo simulations of percolating systems. Then one should compare with experiments (17)

The Infinite Set of Exponents

Suppose that one is interested in higher order cumulants of the resistance fluctuations. By analogy with Eq. (7), these would be obtained from

$$\langle \delta^{2n} R \rangle = \langle \delta r_\alpha \rangle 2n \delta^{2n} R$$  \hspace{1cm} (10)

where the subscript $c$ indicates cumulant average. For the hierarchical lattice of Fig. 2, it is easy to generalize the reasoning of Eq. (8) to obtain

$$\delta^{2n} R = 2 \left[ 1 + \frac{1}{2} \right] \delta^{2n} R$$  \hspace{1cm} (11)

from which one clearly sees that each $\delta^{2n} R$ is controlled by a different exponent. (We dropped the brackets and subscript c for short.)

Eqs. (10) and (11) define an infinite set of measurable $(14, 15)$ exponents to which belong the fractal dimension of the conducting bonds $D(n)$, the resistance exponent $\gamma(n)$ and the noise exponent $\gamma(n)$. It turns out that new is related to the so-called correlation length exponent for reasons that we cannot go into here (8, 14). A note of caution: It has also been shown (18), that negative cumulants do not exist for percolation, while Eq. (12) says they do; one should thus beware that even qualitative information drawn from hierarchical lattices may sometimes be wrong.

To see the generality of the result that each $\delta^{2n} R$ is controlled by a different exponent, note that Eq. (10) defines the $\delta^{2n} R$ as moments of the current distribution in the fractal. Even though this current distribution is not random, it is useful to consider it as being given by a probability distribution. It is then a general result of probability theory (19) that the logarithm of the moments of a probability distribution is a convex decreasing function of $n$. This means that if the moments scale as a power of $t$, the corresponding exponents are a convex decreasing function of $n$. This does not forbid that all exponents be equal (the degenerate case) or that all exponents larger than a certain value be different, but the most general case is that they are different.

The Current Distribution

We are naturally lead to ask what are the general features of the current distribution in fractal lattices. Note in passing that it is known that a probability distribution for a variable which varies on a finite interval (such as the current here which varies between 0 and 1) may be determined from its positive integer moments (20). This is clearly a satisfying result since in our problem these are the measurable quantities. The fact that integer moments suffice to reconstruct the probability distribution, plus a few other results, also suffice to develop deep analogies between the infinite set of exponents described here and the infinite set of irrelevant exponents in critical phenomena (21).

Coming back to our problem, we use a more familiar technique to reconstruct the current distribution from the moments $\langle 1 \rangle$. First, we define the probability distribution $P(C)$,$$

\int [d^2 \gamma] \gamma^{2n} = \sum \frac{\delta^{2n} R}{2} = \left[ 1 + \frac{1}{2n} \right] \sum \delta^{2n} R$$  \hspace{1cm} (12)

where we also used Eq. (11). Then, note that if we change variables to $\gamma = \ln(C)$ in the left-hand
side of Eq. (12), it may be rewritten in the form

\[
\left\{ \begin{array}{l}
\frac{\partial y}{\partial \tau} e^{-T} y(x, \tau) = \tau x(\tau) - c(\tau)
\end{array} \right.
\]

where \(F(\tau) xy = F(\tau) x y\). The distribution \(F(\tau) x y\) may then be found by using the inverse Laplace transform

\[
F(y) = \int_{-\frac{i}{\tau}}^{\frac{i}{\tau}} \exp(\frac{T}{\tau} x(\tau) - \tau c(\tau)) d\tau
\]

Using, from Eq. (12), the explicit expression for \(r^2(l^2)\) and the binomial expansion for the quantity in square brackets, one immediately finds

\[
N = \sum_{n=0}^{\infty} \frac{2n^2 + 1}{n!} C(1 - C)^n
\]

where Dirac's delta function appears. Eq. (15) is the result (16) that one can also obtain directly.

There is a simpler asymptotic form which one may obtain for \(F(y, \tau) \to \infty\) in the limit \(y/\tau \to \infty\). To find it, it suffices to use the Stirling approximation for the factorials in Eq. (16) and more generally, to use the saddle point approximation in Eq. (14). In this approximation, one replaces \(\exp(-n^2/2)\) by \(\sqrt{\frac{2\pi}{n}}\), which is an analytic function of \(y\) on the real axis in convex and increasing. Defining \(f(y) = y/\tau\), one obtains

\[
f(f) = \frac{C(n)}{1 - C} \frac{C(n-1)}{C(n-2) + \tau}
\]

with \(C(n)\) a function of \(n\) determined from the equation,

\[
dr(\tau) f(\tau) = \frac{1}{\tau}
\]

Eq. (16) is the only asymptotic form \(C(\tau) \to \infty\). To a distribution can have, in the saddle point approximation, if its moments are all scaling with a different exponent.\[f(\tau) \to \tau^\alpha\] Note, however, that one does not need to be in the asymptotic limit to have moments which scale with different exponents. In fact, in all cases we know of, and as has been noted by others,\[f(\tau) \to \tau^\alpha\] the exponents for the moments converge to their asymptotic form much faster than the distribution itself.

Eqs. (16) and (17) were found first in Refs. 16, 23. The quantity \(\tau \to \infty\) for fixed \(\tau\) is interpreted as the fractal dimension of bonds whose current scales as \(f^2\). \(f(\tau)\) does not however contain more information than \(r^2(l^2)\). Furthermore, in disordered systems, one encounters cases where \(f(\tau)\) is not well defined and hence the interpretation just given does not always hold. Infinite sets of exponents have arisen in many fields related to fractals. They have been first mentioned in 1977.

CONCLUSION

Exactly self-similar structures have no physical meaning by themselves, but they provide insight into the dilation invariance which is broken by the renormalization group to the class of disordered systems to which belongs, as an important example, the percolation problem. In the disordered systems, it is the probability distributions of physical quantities which obey this dilation invariance (or symmetry) but most of the physics and even some of the more sophisticated calculational techniques may be understood by calculations on "exact" fractals.

Acknowledgments

I am indebted to many colleagues who have shared their ideas on these problems over the years: G. Albinet, D. Breton, O. J. Plucene, P. Fourcade, R. Espanol, and C. Fannes. I would like to thank P. Fourcade for numerous suggestions on the present work. Support was provided by NERC (Canada) through an individual operating grant 241 and through its program "Ataliche de recherche" and by the National Science Foundation (NSF) under grant DMR-85-144-44, administered by the Materials Science Center (Grant #8953).

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