Multifractal Analysis in the Circle Map: Analogies with Critical Phenomena

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Over the last few years, the description of the scaling properties of strange attractors has focused on the scaling properties of the closest return-distances as a function of time scale. Each moment of these return-distances is characterized by a different exponent. Also, a universality class characterizes the whole set of exponents. In this paper, it is shown that analogies with critical phenomena suggest that not only exponents but also amplitude ratios, as well as crossover functions, are universal quantities. These should also be as easily accessible experimentally as the exponents themselves. The special case of circle maps at, or near, the golden-mean winding number is discussed.

1. INTRODUCTION

The transition to chaos via quasiperiodicity is one of the problems of dynamical systems whose understanding is facilitated by analogies with thermodynamic critical phenomena. In particular, there exists a renormalization group for maps of the circle which describes the universal properties of the transition for irrational mean winding-numbers.[1,2] More recently, the strange attractor, which is an experimentally accessible quantity [3], has been characterized by an infinite set of exponents.[4]

Although there exists a renormalization group for this hierarchy of exponents [5], it has not been appreciated that this so-called multifractal approach [4,6] does have phenomenological properties analogous to those of critical phenomena. We report here on these analogies, restricting ourselves to the vicinity of the golden-mean winding-number. Our analysis does point out that the structure of the strange attractor at the critical point, as well as in the crossover region, contains many more experimentally accessible universal quantities than realized before. Renormalization group justifications of the phenomenology that follows will appear elsewhere.

2. THE CRITICAL POINT

For cubic critical-maps of the circle, the critical point is approached by successive rational approximations of the golden-mean. A paradigm for such maps is the sine map

$$\theta_{n+1} = \theta_n + \Omega - (K/2\pi) \sin(2\pi \theta_n). \quad \text{mod}(1)$$

(1)

For rational mean winding-number $F_n/F_{n+1}$ (the Fibonacci sequence) and given $K_1$, one can always find a value of the bare winding-number such that the origin is a periodic wandering point. However, on the critical manifold ($D=0.6066...$, $K=1$) no point is periodic and the starting point for the moments of the closest return-time distance at the coarsening time scale $F_n$
\[ \phi_q(F_n, \theta_1) = \frac{1}{F_{n+1}} \sum_{i=1}^{F_{n+1}} | \theta_{i+1} - \theta_i |^q, \quad 1 \leq i \leq F_{n+1} \]  

(2)

is arbitrary. It has been shown in [4] that this function scales with \( F_n \) as a power law with a non-trivial exponent \( \tau(q) - 1 \) when the starting point is the origin, but the same result holds for an arbitrary starting point. For reasons similar to the ones exposed in [7] we consider from now on the positive integer \( q \).

Pictorially, one can visualize Eq. (2) as taking the \( q \)-th moment of the "distances" between closest points of the trajectory around the circle. One can however show that, for irrational winding-numbers, one can never cover the whole circle by drawing arcs between these closest return-points. We then compute average values by considering the starting point as a random variable. This probabilistic point of view is justified by experimental convenience (the "origin" is hard to find) and by the fact that the fluctuations of these moments are critical in the same sense as critical phenomena.

Figure 1 shows the cumulative probability distribution of the \( q = 3 \) moment for two different critical maps (the quantities have been normalized to avoid non-universal factors, as will be discussed later). These and other analogous numerical results suggest that the joint probability distribution for the \( q \)-th moments (properly normalized) is a universal function which does not depend on the map considered. In fact, this joint probability distribution has scaling properties analogous to the ones considered in (7) (Eq. 2 with \( \phi_q \) instead of \( H_q \)). Thus it can be deduced, in particular, that the ratios

\[
A(q,p;a,b) = \frac{a}{b} \left( \frac{\langle \phi_q(\theta,F_n) \phi_p(\theta,F_n) \rangle}{\langle \phi_q(\theta,F_n) \rangle \langle \phi_p(\theta,F_n) \rangle} \right)
\]

(3)

\[
\text{Fig. 1 Plot of the cumulative probability distribution of } \phi_q(F_n, \theta_1)/\langle \phi_q(F_n, \theta_1) \rangle \text{ at the critical point for:}
\]

(a) the map \( \theta_{i+1} = \theta_i + 0.2 \sin(2\pi \theta_i) \), with \( n = 1 \);

(b) the sine map with \( n = 7 \); (c) the sine map with \( n = 7 \). Curves (a) and (b) have been translated upwards by 0.4 and 0.2, respectively, for visual convenience.
are universal, in close analogy with amplitude ratios in critical phenomena (\( \ll \)) refers to the cumulant averages). The cumulative probability distributions, as well as the amplitude ratios, have been calculated for a uniform distribution for the starting point (see Table 1). But, it can be shown using the fact \( \phi_q(\theta_1, F) = \phi_q(\theta_1, F^n) \) (\( n \geq 1 \)), that they are independent of the distribution for the starting point for a large class of such distributions. One can therefore conclude that they are measurable quantities, just as the set of \( \tau(q) \).

Table 1. Universal amplitude ratios \( A(q, p; a, b) \) for the sine map (column 1) and for the map \( \theta_{i+1} = \theta_i + \Omega - \pi/2n \sin(2\pi \theta_i) + 0.2 \sin(3\pi \theta_i) \) (column 2)

<table>
<thead>
<tr>
<th>( q ), ( p ), ( a ), ( b )</th>
<th>( A(q, p; a, b) ) (column 1)</th>
<th>( A(q, p; a, b) ) (column 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0 2 0</td>
<td>0.0089</td>
<td>0.0089</td>
</tr>
<tr>
<td>1 2 1 1</td>
<td>0.0187</td>
<td>0.0191</td>
</tr>
<tr>
<td>2 3 1 1</td>
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</tr>
<tr>
<td>2 3 1 2</td>
<td>-0.0072</td>
<td>-0.0069</td>
</tr>
</tbody>
</table>

3. APPROACH TO THE CRITICAL POINT

In the subcritical region, where the map is analytically conjugate to a rotation, the crossover of the critical \( f(a) \) spectrum has been studied by Arneodo and Holtschneider [8] in terms of scaling functions for the exponents \( \tau(q) \). From the point of view of critical phenomena, it is more natural to formulate the problem (9) in terms of scaling functions for the \( q \)-th moments:

\[
\frac{1}{T} \sum_{n=1}^{T} [\theta_{i+n} - \theta_i]^q = (\Omega_n - \Omega_c(K))^{(q)} n^{-\nu/y} \chi(t)
\]

where the sum is taken over all points of a periodic orbit with period \( T = F_{n+1} \). \( \Omega_n \) is the bare winding-number for the corresponding orbit, and \( \Omega_c(K) \) depends on \( K \) to stay on the renormalization group eigendirection. \( G_q \) is a universal scaling function of the argument in square brackets, but the scale factor \( A_n \) depends on the map considered.

It should be stressed that, although the arcs between the iterates and their closest return-time neighbor cover the whole circle in the periodic case, the partitioning of the circle does depend on the bare winding-number on a given locking interval (or equivalently, depends on the starting point). The sum (4) then fluctuates from one starting point to another. Furthermore, the period may be infinite (e.g. when \( \Omega_n = \Omega_c(K) \)) even when the map is not critical. A natural generalization of (4) then consists in probing the system at time \( F_k \) shorter than the period, as in (2).

As before, we also adopt a probabilistic point of view and average over bare winding-numbers in a given locking interval. The scaling properties are summarized by (2) of [7] with \( 1-K/\Omega_c \), \( T \), \( F_k \) replacing \( p-p_c \), \( h \), and \( L \), respectively. The replacement of \( F_k \) by \( L \) points clearly to the analog of finite-size scaling. (Note that to recover (4) from the latter equation one must consider a single trajectory instead of an ensemble, set \( F_k = F_n \) and use the following scaling relation [11]:

\[
T = F_{n+1} = (\Omega_n - \Omega_c(K))^{1/y} \chi(A (1-K/\Omega_c)^{\nu/y})
\]

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Fig. 2 Plot of the cumulative probability distribution for $K = 0.97$ and $\Omega = \Omega_c(K)$ for $n = 7, 8, 9, 11$, respectively.

To describe the crossover to non-critical behavior, (e.g. Figure 2 below), it is natural to introduce a notion equivalent to that of a correlation length. Intuitively, this quantity measures the time one has to iterate before feeling a pure rotation. Since there is no fluctuation in the case of a pure rotation, one can define a set of $\xi_q$ by

$$\xi_q = \sum_{k=0}^{F_{n+1}} F_k \left[ \frac{<q_q(F_{k}, \theta_1)^2>}{<q_q(F_{k}, \theta_1)>} \right]^2.$$  \hspace{1cm} (5)

This definition is the analog of the correlation function expression for the thermodynamic correlation length. One can numerically show that all the $\xi_q$ diverge as $\xi \sim (1-K/K_c)^{-v}$ or $T \sim F_{n+1}$ depending on the eigendirection (with $v$ consistent with the value given by the renormalization group analysis [1,2], 0.996). They differ in the scaling regime only by a metric factor: this single "correlation time" marks the crossover between the critical regime and the pure rotation case, where the point wanders around the circle in a trivial way (all arcs are equal).

The role of the "correlation time" $\xi$ is dramatically illustrated in Figure 2 where the cumulative probability distribution $q_q(0, F_k)$ is plotted for $K = 0.97$ and $\Omega = \Omega_c(K)$ when the sine map is conjugate to a rotation with an irrational winding number. When one probes the system at time $F_k$ shorter than $\xi$ the distribution looks critical, but when this time is larger than $\xi$ the distribution, as expected, tends rapidly to the Heaviside function, corresponding to a pure rotation.
4. DISCUSSION AND CONCLUSION

It is well known that the invariant natural measure of maps of the circle at the golden mean critical point, can be characterized by the scaling properties of multifractal moments. One must however choose the starting point of the trajectory from which these moments are calculated. For irrational winding numbers in general, no point is privileged. As expected, the scaling exponents do not depend on this starting point. On the other hand, the amplitudes of the moments depend strongly on this starting point. In this context, it is natural to consider the starting point as random and to summarize the properties of the multifractal moments by a joint probability distribution for the values of these moments. It is found that this joint probability distribution is independent of the probability distribution of the starting point.

This probabilistic description, extended to the vicinity of a critical point, is akin to that of usual critical phenomena. As expected then, the joint probability distribution for the multifractal moments is a generalized homogeneous function of its arguments and is universal, apart from metric factors. Amplitude ratios are then universal and should be as accessible experimentally as exponents. The analogs of both the correlation length and finite size scaling appear naturally.

This new approach should facilitate the measurement of the universal properties (above those of the exponents) of the critical trajectory, because it handles the arbitrariness of the starting point in a natural way. We have also shown that it can be extended to describe the whole crossover region.

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REFERENCES

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