INTRODUCTION

Consider a macroscopic mixture of insulating and conducting material, such as Ag in KCl, or an aluminum sheet with holes in it. At low concentrations of the conductor, the mixture is insulating. As the concentration $p$ is increased, one reaches a "critical" value $p_c$, the so-called percolation threshold, at which the mixture becomes conducting. Near the percolation threshold, there is a correlation length $\xi$ which diverges as $(p-p_c)^{-1}$. On length scales less than the correlation length $\xi$, the structure of the percolating clusters is fractal while on larger length scales it is (on average) Euclidean. Because of the diverging correlation length, most physical quantities near the percolation threshold scale as a power law of the correlation length. More importantly, this power law is "universal", i.e. it is independent of the particular metal and insulator chosen and of many of the details of fabrication of the composite. For simplicity, we will thus consider a lattice model where bonds are occupied by a resistor with probability $p$, and by an insulator with probability $1-p$.

We are interested in the $1/f$ noise of such a mixture. We will assume that $1/f$ noise simply comes from resistance fluctuations. There are then two cases of interest: a) If $1/f$ fluctuations are uncorrelated for distances larger than the lattice spacing, we can assume that every resistor is fluctuating independently. The overall spectrum of fluctuations is the same as that of the individual resistors and the only question left is how does the magnitude of $1/f$
noise diverge close to the percolation threshold. b) If 1/f noise is correlated over distances larger than the lattice spacing, an additional question arises: how is the spectrum of fluctuations modified by the fractal structure of the lattice? The answer to the latter question clearly depends on the noise mechanism. For definiteness, we will discuss the diffusion noise case. It is known that the statistical self-similarity of the percolating cluster leads to anomalous diffusion, which, in turn, modifies the frequency dependence of the spectrum. In the following we will briefly discuss both the limits a) and b). Most of the results have been published elsewhere so we will not go into any mathematical detail of the derivation.

**DIVERSION OF 1/f NOISE CLOSE TO THE PERCOLATION THRESHOLD**

In this first case, we assume that 1/f noise is correlated over distances much shorter than the lattice spacing. Experimentally, this should be the case in many situations of interest since the correlation length of 1/f noise is usually extremely short. In the lattice model then, each resistance fluctuates in time around an average value \( r \), independently from the other resistances. In other words, if \( \alpha \) and \( \beta \) are two of the resistances of the system,

\[
\begin{align*}
\delta r_\alpha &= r + \delta r_\alpha \\
\langle \delta r_\alpha \rangle &= 0 \\
\langle \delta r_\alpha \delta r_\beta \rangle &= r^2(\omega) \delta_{\alpha\beta}
\end{align*}
\]

where the brackets stand for ensemble average (or in practice time average). The last \( \delta \) symbol is Kronecker’s, i.e. is equal to 1 when \( \alpha \) and \( \beta \) are identical and 0 otherwise. Note that we assume that each of the elementary fluctuating resistance has an identical spectrum \( r^2(\omega) \), but the exact frequency \( \omega \) dependence of this spectrum is not important for the following.

To compute the magnitude of the resistance noise, we proceed as follows. First recall that the total resistance may be calculated from

\[
E l^2 = \sum_\alpha \frac{r}\alpha l_\alpha^2 = \frac{r}{\alpha} \frac{l_\alpha^2}{\alpha}
\]

(2)
where \( I \) is the total input current and \( i_\alpha \) is the current in branch \( \alpha \). To compute the overall resistance fluctuations of the circuit, we resort to theorems used before in the context of 1/f noise by the Dutch school, among others. Cohn's theorem, which is a direct consequence of Tellegen's theorem, shows that to linear order in the fluctuations\(^3\)

\[
\delta R I^2 = \sum_\alpha \delta \rho_\alpha i_\alpha^2.
\]

Normalizing the input current to unity, and using the model of Eqs. (1a-c), one obtains

\[
\langle \delta R(t)\delta R(0) \rangle = \rho^2(t) \sum_\alpha \Gamma_\alpha t^4.
\]

We then ask the following question: How does the magnitude of 1/f noise diverge near the percolation threshold, and is this divergence governed by a new exponent or by an exponent related to previously defined ones?\(^5-8\) Let us recall that the geometrical properties of percolating clusters\(^9\) are usually determined by two exponents, \( \beta \) and \( \nu \): \( \nu \) is the correlation length defined before and \( \beta/\nu \) is related to the so-called fractal dimension \( D \) of the infinite cluster by \( D = d - \beta/\nu \). The DC electrical resistance on the other hand diverges as \((p-p_c)^{-\kappa}\) for \( p \geq p_c \).

What was found\(^5-8\) is that the noise diverges as \((p-p_c)^{-\kappa}\) for \( p \geq p_c \), with \( \kappa \) a new exponent different from any of the previously defined exponents for percolation\(^5,7,10,11\): \( \kappa = 1.12^{\pm}0.03, 1.56^{\pm}0.13, 1.57^{\pm}0.14, 1.8^{\pm}0.1, \) and \( 2 \) respectively for \( d = 2, 3, 4, 5, \) and \( 6 \).

More generally, if one is interested in higher order cumulants of the resistance fluctuations it was found that each of these cumulants diverges with a new exponent. By analogy with Eq. (4), the cumulants are obtained from

\[
\langle \delta R^n \rangle_c = \langle \delta R^n \rangle_c \sum_\alpha \Gamma_\alpha t^{2n}.
\]

where the subscript \( c \) indicates cumulant average. The actual numerical results are obtained from simulations on samples of size \( L \ll \xi \) from which one extracts a power law dependence on \( L \).
Eq. (3) leads to an infinite set of measurable exponents to which belong the fractal dimension of the conducting bonds ($n=0$), the resistance exponent ($n=1$) and the noise exponent ($n=2$). It turns out that $n=0$ is related to the correlation length exponent $\nu$ for reasons that we cannot go into here. The exponents describing the dependence on $p-p_c$ are obtained from the $x_n$ through finite size scaling.

The existence of an infinite set of non-trivially related exponents was discovered in many different fields involving fractals; from turbulence to diffusion limited aggregation and dynamical systems. This general phenomenon has now come to be known under the name "multifractals". The present case, to our knowledge, the only one where the infinite set of exponents has direct experimental relevance for macroscopic measurements. Note further that if all the quantities appearing in Eq. (3) are known, they suffice to determine completely the current distribution and hence all the quantities of interest. In other words, we do not need to consider $n$ non-integer.

Before comparing with experiment, we should note an additional complication: In the mapping from the real system to a lattice, the best model leads to a power law distribution for the conductance of the bonds and for the magnitude of their $1/f$ noise. More specifically, one can distinguish at least two simple models: the "random-void," with insulating holes in a conducting matrix, and the "inverted random-void," with conducting holes and insulating matrix. The predictions for the exponents when these "continuum corrections" are taken into account are: $v = x/t \approx 3.2$ and 2.1 in the random-void model, in two and three dimensions respectively, while in the inverted random-void model, $v = x/t \approx 0.87$, and 2.4 in two and three dimensions. These predictions are for the "nodes-links-blobs" model of percolation clusters and hence only approximate. They also differ from some of the predictions of Ref. 18.

Experimentally, the above general ideas have been tested on actual networks of carbon resistors by Giraud et al. The thermal response of films is also related to the exponent $x$ and is being studied by Dubson et al. In composites, one measures the magnitude of the noise as a
function of the resistance instead of as a function of $p-p_c$, which is not easily accessible experimentally. In other words, one obtains directly the exponent $w$. Given the uncertainties of the nodes-links-blobs model for continuum corrections as well as the experimental uncertainties, the following experiments may be interpreted as in agreement with theory. \cite{21} Garfunkel and Weissman\cite{22} find, for sandblasted Al, Cr and In films $w \approx 3.4$ to 5. Given the small range of resistances measured, and the fact that inhomogeneities tend to increase the exponent, this is probably consistent with the random-void model in $d = 2$, $w \approx 3.2$. Rudman et al.\cite{23} worked with mixtures of Ag-Pt alloy in insulating tetrafluoroethylene. Far from $p_c$ they find $w \approx 1$, a result which can be explained by effective medium theory\cite{6}. In the critical regime, close to the transition, they obtain $w \approx 3$. In this case, the appropriate model is the inverted random-void in $d = 3$, which gives $w \approx 2.4$. Finally, Octavio et al.\cite{24} found $w = 0.9^{+0.2}_{-0.1}$ for evaporated Ag films, in agreement with the $d = 2$, inverted random-void model, $w = 0.87^{+0.03}$.

Finally, the following experiments seem to disagree with theory: Koch et al.\cite{25} find $w = 2.1^{+0.1}_{-0.1}$ for $d = 2$ ion milled gold films at room temperature, basically the same value $w = 2.1^{+0.2}_{-0.2}$ as Octavio et al.\cite{24} for ion-milled Ag films at 77 K. Similar results were also obtained in 1969 by Williams and Burdett\cite{26} for evaporated gold films. These authors interpreted their results as coming from noisy tunneling or hopping conduction. Chen and Chou\cite{27} in $d = 3$ mixtures of carbon and wax find $w = 1.7^{+0.2}_{-0.2}$, a result which they also try to interpret as tunneling noise. The results of Mantese and Webb\cite{28} for Pt-Al$_2$O$_3$ and Mo-Al$_2$O$_3$ composite $d = 2$ films were also shown\cite{29} to arise from noisy tunneling conduction in the insulator, and hence are beyond the scope of the present theory. The problem of additional noise mechanisms arising from the insulator has been partially addressed before\cite{28,29,30} but it deserves more attention.

**DIFFUSION NOISE**

Consider the case where the resistance noise is correlated over distances larger than the lattice spacing. This may arise when the noise is diffusive in origin: For example, the resistance of Niobium films can be modulated by hydrogen impurity diffusion\cite{31} or the resistance of superconducting films close to the transition
temperature may be modulated by temperature diffusion. This type of noise mechanism has been proposed more than 35 years ago, but it is only recently that one has obtained experimental evidence that in certain specific cases it is actually the origin of the noise. In these cases, the power spectrum has several power law regimes, none of which is 1/f.

The system we consider is a portion of a large sample. The noise is measured across two planar electrodes of surface area $L^{d-1}$, separated by a distance $L$. The resistance of each branch $\alpha$ fluctuates because of the fluctuations of some quantity $\delta n$ which obeys a diffusion equation. Using Cohn’s theorem, Eq. (3), we have that,

$$S(\omega) = \sum \frac{\partial R}{\partial n} \frac{i^2}{\sigma n} \left[ \frac{\delta R(\omega)\delta R(-\omega)}{\delta n} \right]^2$$

where brackets refer to ensemble (or thermal) averages for the diffusive process. When the diffusion length of the variable $\delta n$ is less than the length of a branch $\alpha$, the quantity in brackets reduces to a delta function and one recovers Eq. (4) of the previous section. In the opposite limit of interest here, assume that $\langle \delta r/\delta n \rangle$ and $r$ are independent of the conducting branch. Let us further average over disorder and over the ensemble corresponding to the diffusing variable. It is then perfectly justified to take $i^2$ and $\delta n$ as independent. If we first restrict ourselves to diffusion lengths larger than the percolation correlation length, we may also neglect the correlations between currents in different branches. Then Eq. (7) reduces to

$$S(\omega) = \int_0^\infty dt \cos(\omega t) \frac{\langle \Delta N(t) \Delta N(t) \rangle}{N^2}$$

where $N^2$ is the square of the number of sites enclosed within the electrode, and $\Delta N(t)$ is the fluctuation of the diffusing quantity averaged over the sites enclosed within the electrodes at time $t$. Because of the last assumption made above, Eq. (8) does not apply at very high frequencies. We consider first the case where both the conduction and the resistance fluctuations due to diffusion occur along the percolating cluster. Henceforth, we shall concentrate on the high and low frequency regimes of the power spectrum where the details of the geometrical shape of the region of measurement is
irrelevant apart of the existence of typical length scale $L$.

To understand the simplest physical derivation of the results\(^3\), recall first that $\langle \Delta N(t)\Delta N(0) \rangle / N^2$ can be considered as $\langle \Delta N^2(0) \rangle / N^2$ times the conditional probability that a random walker starting at any site lying between the electrodes will visit the system again at a time $t$ later. We restrict ourselves to the case of experimental interest, i.e. when the system size $L$ is much larger than the percolation correlation length $\xi \approx (p-p_c)^{-\nu}$. ($L \gg \xi$) Within a correlation length, the number of sites scales as $\xi^D$, where $D = d - \beta/\nu$ is the fractal dimension. Within a region of size $L$, there are $(L/\xi)^d$ such regions. Since $\langle \Delta N^2(0) \rangle$ scales as the number of sites on the percolating cluster, it is proportional to $\xi^D (L/\xi)^d$. On the other hand, it is known that as long as the diffusion length is larger than $\xi$, the diffusion is "normal" but the diffusion constant $D(p)$ scales\(^4\) as $\xi^{\beta/\nu - t/\nu}$. There are then three regimes to consider.

1. $u \ll D(p)L^{-2}$

The diffusion length is then much larger than the system size. When $d < 2$, the random walk is recurrent and the conditional probability of return to the original domain is proportional to $L^d D(p) L^{-d/2}$ so that

$$\begin{align*}
S(\omega) & \approx \frac{\xi^{\beta/\nu}}{[D(p)]^{d/2}} u^{d/2 - 1} \approx (p-p_c)^{-\beta / (\beta + d/2)} (\beta - t) u^{d/2 - 1} \quad \text{if } d < 2.
\end{align*}$$

On the other hand, if $d > 2$, the walk is not recurrent so that the probability of return to the origin decays in a time of the order of the time taken to cross a system of size $L$, i.e. $L^2 / D(p)$. In other words, the quantity $\int_0^\infty \Delta N(t) dt$ is finite and proportional to $\Delta N(0) L^2 / D(p)$. This leads to the spectrum,

$$\begin{align*}
S(\omega) & \approx \frac{\xi^{\beta/\nu}}{[D(p)]} L^{2-d} \approx (p-p_c)^{t} L^{2-d} \quad \text{if } d > 2.
\end{align*}$$

2. $D(p)L^{-2} \ll u \ll D(p)\xi^{-2}$

One is in the high frequency Euclidean regime. The diffusion length is however still larger than the percolation correlation length so that the frequency dependence does not reflect any anomalous diffusion. The frequency dependence of the spectrum is obtained from
the Lax-Hengert argument.\(^{34}\) The time is so short that only the random walkers near the boundary of the system have time to escape, i.e.

\[
\Delta N(t) = \Delta N(0) \left[ 1 - Q (D(t) t)^{1/2} S/N \right]
\]  

(11)

where \(S\) is the surface of the system, \(D(t)t^{1/2}\) is the distance traveled by random walkers in one dimension in time \(t\), and \(Q\) is a numerical factor of order 1/2 taking into account the fraction of random walkers which actually exit the system. With \(S \approx \xi^{d-1} (L/\xi)^{d-1}\) and our previous results, this leads, after Fourier transformation, to

\[
S(\omega) \approx \xi^{\beta/\nu} \frac{D(t)^{1/2}}{L^{d+1}} \omega^{3/2} \approx \frac{\langle p-p_c \rangle^{1/2 - \beta/2}}{L^{d+1} \omega^{3/2}}
\]  

(12)

3. \(D(t)^{1/2} \xi^{-(2+\theta)} \ll \omega \ll \omega_c\)

Here \(\omega_c\) is the highest frequency where the diffusion approximation is valid. Note that \(\theta\) is defined by, \(D(t) \equiv D_1^{1+\theta/2} \xi^{-\theta}\) or \(\theta = \mu - \beta/\nu\). Suppose that the Lax-Hengert argument\(^{34}\) still applies. Since the diffusion length is smaller than the percolation correlation length, the diffusion is anomalous\(^4\) so that instead of \(D(t)^{1/2}\) we have \(D_1^{1/2} \xi^{(2+\theta)}\). This immediately leads to,

\[
S(\omega) \approx \xi^{\beta/\nu} \frac{D_1^{1/2}}{L^{d+1}} \omega^{-(3+\theta)/(2+\theta)} \approx \frac{\langle p-p_c \rangle^{\beta/\nu}}{L^{d+1}} \omega^{-(3+\theta)/(2+\theta)}
\]  

(13)

One of the nice features of the last result is that the spectrum is sensitive to the fractal aspects of the problem even though the macroscopic probes are at a length scale much larger than the percolation correlation length.\(^{3,4}\). Eq. (13) could be observed in light scattering experiments for diffusion on the infinite cluster (no finite clusters) since in that case, Eq. (8) and hence the Lax-Hengert argument apply.

In the case of a noise experiment however, the diffusion length at high frequencies is smaller than the percolation correlation length and hence we cannot assume that \(i_q^2\) and \(i_{q'}^2\) are uncorrelated. This means that Eq. (7) does not reduce to Eq. (8). In this regime, there is
a loss of correlation not only when random walkers go out of the region of observation, a surface effect, there is also a loss of correlation within every correlation length of the system, a volume effect. Indeed, within a correlation length, the correlation between two weighting factors \( i^2 \) separated by a distance \( \xi \) decays as \( \frac{1}{\xi^{2(D_B + x_1)}} \), where (see Eq. (3)) \( x_1 = \frac{t}{v} + (2 - d) \) and \( D_B \) is the fractal dimension of the conducting backbone. Following this through, we obtain that in this case,

\[
S(\omega) \approx \frac{\xi^{D + D_B + 2x_1}}{L^d} \frac{\pi}{\pi_x^{x_1}(\pi x_1)^2} \frac{\zeta_{-1}(D + D_B + 2x_1)}{(D - x_1)^2}
\]

(C. 4)

The frequency dependence is close to \( \omega^{-0.5} \) in \( d = 2 \) or \( d = 3 \). At this point, Eq. (C.4) is only a preliminary result. We are currently exploring this effect in more detail.

Ref. 3 contains numerous other cases that we do not have space to discuss here. In particular one can obtain the modifications to the Nyquist–Johnson noise. Experimentally, the above predictions could be verified for example in percolating networks made of material where hydrogen diffusion leads to resistance noise. In the case where the diffusion is on a Euclidean network while the conduction is along a percolating network, the spectral shape is identical to the standard one \( ^2, ^3 \) as long as the diffusion length is larger than the percolation correlation length. In the opposite limit, one can compute the spectrum following arguments similar to those which lead to Eq. (C.4).

**CONCLUSION**

While the divergence of \( 1/f \) noise near the percolation threshold in metal-insulator mixtures may be understood from the theory presented in the first part of this talk, some experiments are still not understood. Hopefully, going beyond the modes-links-blobs model for continuum corrections and taking into account noisy conduction through the insulator will help explaining the results. On the theoretical side, the study of this problem has lead naturally to a physically simple example of multifractals. In this context higher-order cumulants of the noise are interesting. Noise in nonlinear circuits has also recently been a subject of theoretical
Finally, no experiment has been done yet, to our knowledge, on the second problem discussed here, i.e. diffusion noise in percolating networks.

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