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OBSERVABLE INFINITE SETS OF EXPONENTS IN MULTIFRACTALS AND IN CRITICAL PHENOMENA: THE ROLE OF SYMMETRY

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ABSTRACT

Continuous sets of exponents appear naturally in many fields related to fractals. Using percolation as a primary example, analogies and differences between a specific field theory for "multicritical" behavior and one for "multifractal" behavior can then be discussed. It is shown that each exponent is associated with an operator of different symmetry. This seems to be the deeper reason behind the observability of these exponents even in situations where they would otherwise be called irrelevant. Further similarities and differences between multifractals and critical phenomena are discussed. For example, in the context of multifractals the concept of a "second renormalization group" and that of "dominant exponents" are introduced.

1. INTRODUCTION

Thanks to the renormalization group, and to the geometrical concepts of fractals, much progress has occured in our understanding of the Physics of states of matter with quenched disorder. For example, the concepts of phase transitions and scaling provide us with a deeper understanding of the behavior of physical properties in the vicinity of the critical concentration at which a random mixture of metal and insulator changes from metallic to insulating: the so-called percolation problem [1]). Near that critical concentration, or threshold, at which the transition occurs, fluctuations become enormous, and a single length, the correlation length, dominates the problem. Just as in critical phenomena, physical properties obey the principles of scaling and universality, which in turn are ultimately understood through the concepts of the renormalization group. Other problems include, for example, localization and diffusion limited aggregation, which we will not discuss anymore in the present paper, even though there are many analogies with percolation.

The renormalization group and critical phenomena have thus provided, over the years, the key concepts which allow us to understand problems such as percolation. That lore was challenged relatively recently by the appearance of a measurable infinite set of exponents in various fields. Problems where such infinite sets occur are collectively known as *multifractal* problems [2]), even though some of these problems have a quite different nature. We discuss here the problem of electrical properties of percolating networks [3]), [4]), which is most closely related to problems such as localization and diffusion limited aggregation, where the multifractal properties basically arise form the properties of the solution of Laplace's equation on a fractal network.

To be more specific, let us describe how the infinite set of exponents appears in percolation through the problem of noise [3]). Suppose that the conducting resistors of a percolating network are fluctuating independently in time. The total resistance of a given network is then a random variable in time whose cumulants depend on those of each component resistor. The cumulants of a given order are assumed to be the same for all component resistors. The cumulants of the total resistance R are, in principle, accessible experimentally, and measurements of the second cumulant, corresponding to 1/f noise, have actually been performed [5]. Theoretically, for systems of finite size L at bulk criticality, one finds [3], [4] for the scaling of the cumulants of order n,

$$C_R^n(L) = \langle \delta R^n \rangle_{cum} \equiv \langle M_n \rangle \sim L^{-x_n} \quad ;$$

$$M_n \equiv v_n \sum_\alpha i_\alpha^{2n} \quad , \tag{1}$$

where v_n is the amplitude of the n'th cumulant of the elementary resistance fluctuations, i is the current that flows in branch α of the time averaged network when the total input current is unity and where the brackets refer to time *and* disorder averages. Each exponent x_n is different and is not a simple linear function of n -as commonly occurs in critical phenomena under the name "gap scaling" [6]-. Such an infinite set of exponents also arises in the other analogous problems mentioned above. In all cases, one is assigning to parts of a fractal network a weight (measure) which is obtained from the solution of Laplace's equation. Here that weight is the power dissipated in a bond.

Why is the existence of the function x_n a challenge [7] for critical phenomena? Even though infinite sets of exponents, such as crossover exponents, were calculated early after the introduction of the renormalization group in critical phenomena, [8], [9] attention is usually focused on a few relevant exponents. This focus on a few exponents is justified since observable quantities in general couple to many renormalization group eigenoperators, including the most relevant ones, which eventually dominate their behavior. In fact, the contrast is even sharper when one realizes that in some of the contexts where infinite sets of exponents were discovered, the definition (1) was extended to real values of n. -In the context of percolation, for sufficiently negative values of n, the M_n do not depend on size as a power law.[10] - We thus have two obvious questions, the first of which is:

A) Do we need an infinite set of exponents which is parametrized by a continuous index? Even though that set is infinite, we believe that it is fundamentally discrete. Indeed, it has been pointed out [11]) that the positive integer values of n suffice to characterize the set. This follows from the Hausdorff- Bernstein reconstruction theorem [12]) because (1) may be interpreted as moments of a distribution probability for the variable i^2, whose range is finite:]0,1]. Pushing the analysis even further, we then consider the integer moments themselves as random variables. This allows us to point out analogies with critical phenomena through a joint probability distribution whose scaling properties are given by, [11])

$$\mathcal{P}(M_1, M_2, \ldots M_\infty, p-p_c, h, L) = \lambda^{x_0+x_1+\ldots+x_\infty}$$

$$\mathcal{P}(M_1/\lambda^{-x_1}, M_2/\lambda^{-x_2}, \ldots M_\infty/\lambda^{-x_\infty}, p-p_c/\lambda^{-1/\nu}, h/\lambda^{-\phi_h/\nu}, L/\lambda) \quad (2)$$

where p is the probability that a site or bond is occupied and h is the analog of magnetic field in percolation [1]).

The formal similarity of Eq. (2) with the scaling properties of the free energy near a critical point immediately suggests [11]) that $\mathcal{P}$ is a universal function of its arguments except for constant scale factors for each of these arguments. Universal amplitude ratios become of interest and should be measured. They have already been calculated in the context of percolation [11]) and in the context of dynamical systems [13]) .

It should be clear from the preceding paragraph that analogies with critical phenomena are useful since they suggest, for multifractals, a much richer class of universal quantities to measure than had been suspected. Nevertheless, one may wonder whether analogies hold in every detail or whether there are differences. [14], [15], [16] In particular we have not yet completely addressed our first remark on the fact that measurable operators in general have a non-zero projection on almost all eigenoperators, and in particular on the most relevant ones, which eventually dominate the scaling behavior.

B) In the case of multifractals, do we have an infinite set of relevant operators? Or if not, why don't observables couple only to the most relevant ones? It is this and related questions that are considered in more detail below. Here, it appears that symmetry and not relevance or irrelevance is the basic reason behind the observability of different exponents. In fact we will see that the exponents $-x_n$ are better characterized by the adjective "dominant" instead of relevant or irrelevant. [15], [16]

Finally, we should note that since in the limit $\ell nL \to \infty$ the $-x_n$ are proportional to the logarithm of moments of a probability distribution, they are a convex non-increasing function of n. In dynamical systems, the Legendre transform of the corresponding function has been interpreted geometrically as a fractal dimension [7] and has thus become widespread, even though one must sometimes beware of the geometrical interpretation [17]. In the context of percolation, that Legendre transform -the so-called $f(\alpha)$ function- may be used to characterize the scaling behavior of the probability distribution for the dissipated power, in the limit where the logarithm of the coherence length goes to infinity. There thus arises another question:

C) Is there the analog of the $f(\alpha)$ function in critical phenomena? The answer to the latter question seems to be: in certain special cases only. This question will not be discussed here. It is the subject of another paper. [18]

2. MULTICRITICAL BEHAVIOR IN CRITICAL PHENOMENA

In certain cases, the infinite set of crossover exponents ϕ_n which arises in phase transitions with a continuous symmetry may be experimentally accessible. This was achieved probably for the first time in 1974 in neutron diffraction experiments on the magnetic structure factor of Erbium [18] It is however only with the work of Cowley and Bruce [19] in 1978, Per Bak [20] in 1980 and Brock et al. [21], [22], [23] in 1986 that the true significance of that work became clear. To be more specific, the set of crossover exponents in question is associated with each possible symmetry-breaking operator for the three-dimensional XY model. It is symmetry in fact which makes everyone of the corresponding operators governed by a different exponent instead of by the most relevant ones. This was realized theoretically very early after the introduction of the renormalization group. [24] Note also that, in the present case, all the symmetry-breaking perturbations, except the first three ones, are irrelevant: Nevertheless, they are, in principle, all observable, even though this is more and more difficult for the higher order symmetry-breaking perturbations.

The XY fixed point is a particularly simple case of a fixed point with a continuous symmetry where the infinite set of symmetry-breaking perturbations can become accessible experimentally.[21], [23] Indeed, the symmetry is O(2) so that harmonics of the angular dependence of the diffraction pattern contain information about the symmetry-breaking operators.[22], [21], [23] The situation is more complicated with a more general continuous symmetry, such as O(n) with $n > 2$ since, in that case, more than one angular variable is necessary, so that simple experimental probes are not easy to devise. The probe used in the case of the XY model would, for $n > 2$, couple to more relevant operators, as in one of the cases discussed by Castellani and Peliti [25].

Infinite sets of *relevant* symmetry-breaking operators near a fixed point with a continuous symmetry also exist, as was probably first discovered in 1976 by Brézin et al. [26]. As mentioned above however, relevance helps the observability of the exponents but is not essential. Wegner in 1980 [27] applied ideas similar to those of Brézin et al. to localization. As discussed by Castellani and Peliti [28], the "multifractal" description of wavefunctions near a localization threshold is not really different from the infinite set of crossover exponents of Wegner. Although the analysis of Wegner also shows clearly analogies with critical phenomena, we discuss below the approach of Park Harris and Lubensky [28] (PHL) to the percolation problem and point out in that context analogies and differences with critical phenomena. [14], [16]

3. MULTIFRACTAL BEHAVIOR ON PERCOLATING NETWORKS

The physical problem of electrical properties of percolating networks has been described in the introduction. In the field theory [28] of PHL for this problem, an n-dimensional replica space, with components labeled α, is introduced to perform the equivalent of the cumulant averages of the noise for a given percolating network. Then, the average over different realizations of the percolating network is performed through an m-dimensional replication of this n dimensional replica space. Each replication is labeled by an additional index β. The m and $n \to 0$ limits are to be taken as usual at the end of the calculation.

The derivation of the PHL field theory is lengthy, but all details may be found in Refs. [29] and [28]. Equations (3.4b),(3.19),(3.20) of Ref. [28] and (3.23 to 3.28b) of Ref. [29] show that, within unimportant constants, the generating function is,

$$G_k(x,x') = \langle \Phi_k(x)\Phi_{-k}(x')\rangle \tag{3}$$

where the brackets refer to traces over the $\Phi_k(x)$ fields with the following action L:

$$L = \int d^dx \; [\; \frac{1}{2} \sum_{k \neq 0} r_k \Phi_k(x)\Phi_{-k}(x) + \nabla\Phi_k(x)\nabla\Phi_{-k}(x) \;] \; \Delta k$$

$$- \frac{1}{3} u_3 \int d^dx \; [\sum_{k_1, k_2, k_1+k_2 \neq 0} \Phi_{k_1}(x)\Phi_{k_2}(x)\Phi_{-k_1-k_2}(x)] \Delta k_1 \Delta k_2 + .. \qquad (4)$$

The k in (3) is anyone of the k Fourier variables whose components are labeled $k_{\alpha\beta}$ in the mn-dimensional replica space. These variables are conjugate to the electrical potentials of the replicated systems. In the limit of geometrical percolation, (i.e. no transport property), the variables $k_{\alpha\beta}$ are all zero. Note that despite the notation, the k are tensors of rank one and not two, as far as rotations in replica space are concerned. For each replica, the components of k can take s discrete values. These values are discrete because one can apply the replica trick only for a finite system. The infinite volume limit $(s \to \infty)$ must then be taken after the limit $nm \to 0$. There are $(s^{nm}-1)$ fields $\Phi_k(x)$ at each spatial point x (the k = 0 case is omitted). u_3 is a coupling constant whose k dependence is neglected while

$$r_k = r_0 + \sum_{m \geq 1} v_m \; k^{2m} \; P^{(m)}(\ldots, \theta_k \; , \ldots) \qquad (5)$$

where $r_0 = (p - p_c)$ with p_c the critical value of the dilution probability p, $k^2 = \sum_{\alpha\beta} k^2_{\alpha\beta}$ while the $P^{(m)}(\ldots,\theta_k \;,\ldots)$ are arbitrary functions of the angular variables θ_k in the replica space. The exact form of these functions depends, in particular, on the statistical properties of the elementary resistance fluctuations. The v_m were defined in Eq. (1) and, to linear order in v_m, homogeneity considerations imply the functional form of Eq. (5). The result (5) differs from that of PHL and also in what follows we differ somewhat with them in our approach to the problem, even though the final results for the exponents are identical.

The cumulants $C_R^{\ell}(x,x')$ of the resistance noise measured between two points x and x' may be obtained from Eq. (3) through derivatives of the form

$$C_R^{s}(x,x')\chi_p(x,x') = 2^s(-1)^s s! \; v_s k^{-2s} \left.\frac{\partial G_k(x,x')}{\partial v_s}\right|_{v_s = 0} \quad \forall s \tag{6}$$

where χ_p is the susceptibility function for percolation. Clearly the v_s play the role of fields while the k play the role of operators.

The exponents are as usual obtained from the renormalization group recursion relations. The relation for u_3 is the same as for pure geometrical percolation (i.e. as if all the k and v_s were vanishing.) u_3 in the recursion relation for r_k may then be taken at its fixed point value. To one loop order in the $c = 6 - d$ expansion, that recursion relation for the r_k reads,

$$r'_k[v_s P^{(s)}(k_{\alpha\beta})] = (2-\eta_p)v_s P^{(s)}(k_{\alpha\beta}) - g\,\tilde{\Sigma}_k(v_s P^{(s)}(k_{\alpha\beta})) \tag{7}$$

where $g = c/7$ is proportional to u_3^{*2} and where the self-energy may be rearranged as follows by using the Schwinger representation of the propagators and expanding in v_s for $s \geq 2$:

$$\tilde{\Sigma}_k = \frac{1}{2}\int_0^\infty du \int_0^\infty dt\, \exp[-(u+t)(1+r_0)]\, \left(\frac{t}{u+t}\right)^{2s} u$$

$$\exp\left[\frac{u+t}{2v_1}\, t^{-2}\Delta\right] \sum_{s\geq 2} v_s\, k^{2s}\, P^{(s)}(\ldots,\theta_k,\ldots)\,. \tag{8}$$

Δ is the Laplacian operator for a space of dimension nm. Clearly (7) must be interpreted as an eigenvalue equation in the space of polynomials of $k_{\alpha\beta}$. That equation, as it stands, is not diagonalizable in the space of eigenfunctions of the nm-dimensional Laplace's equation [30] (harmonic polynomials). However, we note that the fact that the system is finite before the limit $nm \to 0$ is taken implies that $2\pi/\Delta V_{max} \le k \le 2\pi/\Delta V_{min}$. Hence, on that finite domain, we can expand in the following complete set of functions,

$$\frac{J_{2s-(nm-2)/2}\left(\zeta_l \frac{\Delta V_{min}}{2\pi} k\right)}{\left[\zeta_l \frac{\Delta V_{min}}{2\pi} k\right]^{(nm-2)/2}} Y_{2s}(\ldots,\theta_k,\ldots) \tag{9}$$

where J_ν is a Bessel function of order ν, ζ_l is the set of zeros of that function, and Y_{2s} are the generalized spherical harmonics derived from the harmonic polynomials of order 2s (we do not explicitly write down the other indices). The Laplacian acting on these functions has eigenvalue $-\left[\zeta_l \frac{\Delta V_{min}}{2\pi}\right]^2$. Since $\Delta V_{min} = \sigma_0^{-1} i_{min}$ where $v_1 = \sigma_0^{-1}$ is the resistance of an elementary bond, then at constant current the quantity $\Delta V_{min}^2 / v_1$ vanishes in the limit $v_1 = 0$ (where the derivatives in (6) are evaluated). In this limit then, the $P^{(s)}(k_{\alpha\beta})$ are eigenvectors with the same eigenvalue as the harmonic polynomials of the same degree. The eigenvalue depends only on the degree of that polynomial, i.e. not on the angular variables. This greatly helps to give a meaning to the $mn = 0$ limit!

i) Infinite Set of Exponents (multifractal behavior): From the recursion relations for the parameters entering (4) and (5) and standard scaling arguments, one finds as PHL that (for $| x - x' | \ll \xi$)

$$C_R^s(x,x') \sim | x - x' |^{\psi_s/\nu} . \qquad (\psi_s/\nu = -x_s) \tag{10}$$

This infinite set of exponents is not a linear function of s. Each ψ_s appears as the leading exponent corresponding to a given symmetry (See iii below).

ii) <u>Gap Scaling</u>: The scaling of the usual thermodynamic observables is normally trivially obtained from a few exponents only. This is usually refered to as "gap scaling". Gap scaling also occurs in the present case. For example,

$$\left.\frac{\delta G_k^{\ell}(x,x')}{\delta v_s^{\ell}}\right|_{v_s = 0} \sim |\, x - x' \,|^{\ell \psi_s/\nu} \quad . \tag{11}$$

In other words, the exponents describing the scaling of the susceptibilities associated with the basic fields v_s are all linearly related to the basic exponents ψ_s.

iii) <u>Crossover Exponents and Symmetry-Breaking Perturbations:</u> Assuming s large enough that k is a continuous variable, the action (4) with $v_\ell = 0$ is invariant under the *global* transformation of the fields,

$$\Phi'_k = \Phi_{Rk} \tag{12}$$

where R is a rotation of the vector k in the replica space of dimension mn. In other words, the action transforms according to the unit representation of the group O(mn). When $v_\ell \neq 0$, that symmetry is broken since polynomials of higher degree transform like higher dimensional representations of the group O(mn). This is analogous to what happens with the symmetry-breaking fields in the XY model discussed in the previous section. The ψ_s here are analogous to the crossover exponents ϕ_n of the XY model discussed above.

iv) <u>Relevance and Irrelevance:</u> At first sight, the perturbations associated with the v_ℓ are all relevant since all the exponents ψ_ℓ are found to be larger than zero. [28]) There are two important differences however with the XY case of critical phenomena.

a) There is no physical realization that we know of for the lower symmetry fixed point towards which the system rescales when one of the symmetry-breaking perturbations is different from zero. (All physical observables are derivatives evaluated at a zero value of the symmetry-breaking fields v_s: they are crossover exponents associated with the symmetric fixed point.)

b) There is an additional freedom to rescale the operators k at each iteration which allows one to formulate the renormalization group in such a way that only a finite number of operators are relevant! Indeed, for the usual percolation fixed point, the rescaling of the operators Φ is found by choosing that the coefficient of the spatial gradient term in (4) be a constant. Since the recursion relations for u_3 and r_0 are completely independent of k, the geometric percolation fixed point is the usual one. The scale factor for k may be chosen at will. This influences the recursion relations for the v_ℓ and hence the corresponding ψ_ℓ exponents. Since the ψ_ℓ are a decreasing function of ℓ, we may always choose the scaling dimension of k such that only a few of the ψ exponents are positive, without influencing the Physics. These statements can be rephrased in a more physical way by recognizing that the field theory of PHL corresponds to computing the power dissipated between points x and x' when a unit current is injected between these points, whatever the distance between x and x': One could just as well decide to rescale at unit voltage instead of unit current, and this would correspond to multiplying k by a scale factor at each iteration.

Note that the remark of the last paragraph may also be formulated as follows: The analysis that we have done to find the eigenpolynomials of (7) shows that the latter equation is like a "second renormalization group" with operators k and conjugate fields v_s which describes the electrical properties of an object whose (critical) geometrical properties are given by the "first renormalization group", with fields r_0 and u_3 and operators ϕ_k. The first renormalization group has properties totally independent from those of the second while the second is slaved to the first. On the other hand, the rescaling of the operators k in the second renormalization group is arbitrary, and this is fundamentally due to the linearity of Kirchhoff's laws. Instead of talking about relevant or irrelevant exponents, for that second renormalization group it makes more sense to talk about "dominant" exponents since one expects that the observables which are connected to ϕ_m are also coupled to operators which give corrections to scaling.

v) Observability of the Crossover Exponents: The ϕ_n of the XY model discussed in the previous section are not all relevant exponents. In fact, for $n \geq 4$, they correspond to irrelevant operators. [22]) While they are only a subset of all possible irrelevant exponents, they are, however, special because, for increasing values of n, they represent the leading scaling behavior of operators with lower and lower symmetry. It is their symmetry instead of their relevance which seems fundamental for their observability. The same remark applies for the "dominant" exponents $\phi_m/\nu = -x_m$ discussed above.

4. CONCLUSION

Here we have contrasted one case in critical phenomena where an infinite set of exponents is observable, with one case in percolation where a multifractal description has been proposed, to illustrate that multifractals simply arise from the existence of symmetry-breaking operators near a fixed point with a continuous symmetry. In both critical phenomena and percolation, it seems to be the symmetry which is crucial for the observability of an infinite set of crossover exponents, not their relevance or irrelevance.

The multifractal behavior in the case of percolation is associated with a "second renormalization group" to which an additional normalization freedom (e.g. scaling at constant voltage or constant current) is associated. This freedom allows one to arbitrarily shift the crossover exponents (while maintaining the observable quantities unchanged). We propose to call these exponents *dominant* since, even though their value can be shifted, they are trivially related to the *leading scaling behavior* of operators characterized by a given symmetry. The mechanism for not coupling back to more relevant operators is thus the same as in the case of the XY model.

Finally, a purely geometrical interpretation (such as that of Ref. 7) of the Legendre transform ($f(\alpha)$) of the crossover exponents in the critical phenomena case does not necessarily hold. This is discussed further in another paper.[16] Nevertheless, the analogies are in fact so close that they have suggested, for multifractals, whole classes of universal quantities which we have calculated 11)– 13) but which have yet to be measured.

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