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$1/f$ noise in random resistor networks: Fractals and percolating systems

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A general formulation for the spectral noise S_R of random linear resistor networks of arbitrary topology is given. General calculational methods based on Tellegen's theorem are illustrated for one- and two-probe configurations. For self-similar networks, we show the existence of a new exponent b, member of a whole new hierarchy of exponents characterizing the size dependence of the normalized noise spectrum $\mathscr{S}_R = S_R/R^2$. b is shown to lie between the fractal dimension $\bar{d}$ and the resistance exponent $-\beta_L$. b has been calculated for a large class of fractal structures: Sierpiński gaskets, X lattices, von Koch structures, etc. For percolating systems, $\mathscr{S}_R$ is investigated for $p < p_c$ as well as for $p > p_c$. In particular, an anomalous increase of the noise at $p \to p_c^+$ is obtained. A finite-size-scaling function is proposed, and the corresponding exponent b is calculated in mean-field theory.

I. INTRODUCTION

Statistical self-similarity is emerging as an important concept underlying the behavior of disordered systems. In percolation clusters, for example, the fractal dimension has been identified first;[1] it was immediately realized, however, that this quantity and the correlation-length exponent did not suffice for a characterization of all the physical properties of these clusters. Alexander and Orbach[2] and Rammal and Toulouse[3] introduced the spectral dimension $\tilde{d}$ to describe the spectrum of the Laplacian operator which appears in a large variety of linear physical problems. The geometrical property which most influences the spectral dimension is the number of closed loops of the fractal. It is an intrinsic geometrical property[4] independent of the embedding Euclidean space. Another such intrinsic property is the recently introduced[5,6] spreading dimension $\hat{d}$. Intuitively, it is plausible that an infinite number of exponents must be used to characterize a fractal.

In this paper, we show that at least one physically measurable quantity, the magnitude (not the frequency dependence) of the resistance noise spectrum ($1/f$ noise) depends on a new exponent pertaining to fractal lattices. We show that this exponent, b, can be seen as a member of an infinite family of exponents which includes the fractal dimension $\bar{d}$ as well as $\beta_L = (\bar{d}/\tilde{d})(\tilde{d}-2)$. The spreading dimension is the only exponent which apparently does not fit into this family. Physically, the new exponent b comes from a well-known fact in the $1/f$ noise problem:[7-10] the macroscopic mean-square resistance fluctuations are much more sensitive to local inhomogeneities than the square of the macroscopic resistance itself.[8,11,12]

Flicker ($1/f$) noise[7-9] refers to the low-frequency spectrum of excess voltage fluctuations measured when a constant current is applied to a resistor. That spectrum, $S_v(\omega) = \int e^{i\omega t}\langle V(t)V(0)\rangle dt$ (where the angular brackets refer to time average) almost always has a power-law form $\omega^{-\alpha}$, with α close to unity. The origin of this power law has been the subject of innumerable controversies[7-9] and is not the purpose of the present paper. Rather, we use two well-established properties of $1/f$ noise: (a) $1/f$ noise is resistance noise. In other words, a simple application of Ohm's law suggests that if there are voltage fluctuations δV in the presence of a constant current, they are caused by resistance fluctuations, $\delta V = I\,\delta R$. This naive picture is confirmed by the fact that (i) the noise spectrum is proportional to I^2, and that (ii) the resistance fluctuation spectrum which can be inferred from $1/f$-noise experiments can also be directly measured with no applied current from higher-order equilibrium correlation functions.[10] (b) At low frequencies, resistance fluctuations are correlated over microscopic scales only. This has been verified experimentally in many systems[13,14] and most mechanisms suggested for $1/f$ noise (except diffusion) are consistent with this hypothesis.

In this paper, flicker ($1/f$) noise in self-similar and random resistor networks is considered for the first time. Using known properties of both $1/f$ noise and fractals, we study the influence of the geometrical properties of self-similar networks (e.g., percolating clusters) on the magnitude of $1/f$ noise in these structures. The finite-size dependence of the noise allows us to shed new light on self-similar structures as well as percolating systems.

The general calculational tools for resistance fluctuations are given in Sec. II. With the help of Tellegen's theorem, we derive a formula for the spectral noise adapted to the geometry of the structure at hand. Tellegen's theorem is applied first to one-port circuits, then extended to two-port configurations so as to make contact with recent measurements of Vandamme *et al.*[15] and of Weissman *et al.*[16] Within our model, two-port measurements do not, however, yield new information. In Sec. III, we show the existence of a new exponent b characterizing the scaling behavior of the normalized noise $\mathscr{S}_R=S_R/R^2$ with the linear sample size L. Upper and lower bounds are given for b and for the members of the new hierarchy of exponents. The scaling behavior of $\mathscr{S}_R$ with L is shown to persist for an arbitrary one-port configuration. Certain averages of $\mathscr{S}_R$ are also shown to have a scaling behavior. Section IV is devoted to percolation clusters where again a scaling behavior of $\mathscr{S}_R$ is noted with an exponent lying between $-\beta_L$ and $\bar{d}_B$ the fractal dimension of the percolation backbone. The crossover between the fractal behavior ($p \gtrsim p_c$) and the Euclidean regime ($p > p_c$) is also discussed. Section V discusses our results in the light of existing theories and possible experiments on percolating systems.

II. NETWORKS OF ARBITRARY GEOMETRY: GENERAL THEOREMS

In this section we summarize the general calculational methods for S_R. Only purely resistive networks are considered. It is easy to generalize all the results for networks made of complex impedances.

Let us consider a resistor network as a graph made of N nodes (i) and resistances r_{ij} between nodes i and j $(\neq i)$. In the presence of an external current source, the basic circuit equations can be written as usual,

$$\boldsymbol{G}\boldsymbol{V}=\boldsymbol{I}\ , \tag{1}$$

where $\boldsymbol{I}$ denotes the (column) vector $\boldsymbol{I}=(I_1,I_2,\ldots,I_j,\ldots,I_N)^t$ of current sources $\boldsymbol{V}=(V_1,V_2,\ldots,V_j,\ldots,V_N)^t$ refers to the relative voltages at different nodes, and $\boldsymbol{G}$ to the conductance matrix. The matrix elements of $\boldsymbol{G}$ are given by

$$G_{ij}=-g_{ij},\quad G_{ii}=\sum_j g_{ij}\ . \tag{2}$$

Here, $g_{ij}=r_{ij}^{-1}$ denotes the conductance of the branch (ij). In the one-port configuration (two-terminal) where the current is injected at node a and extracted at node b, the vector **I** takes the simple form, $I_a=I$, $I_b=-I$, and $I_\alpha=0$ for $\alpha\neq a,b$. The current pattern in different branches is therefore easy to obtain from the inverse of the conductance matrix $\boldsymbol{G}$: From $\boldsymbol{G}^{-1}$, one extracts the voltages at each node and then the currents $i_{ij}=g_{ij}(V_i-V_j)$ for each branch (ij) (at least one node is at the reference voltage).

We consider the following model. The resistance of each branch (ij) has a small fluctuating part so that r_{ij} is replaced by $r_{ij}+\delta r_{ij}$, where the δr_{ij}'s are uncorrelated random variables with mean zero and covariance

$$\langle \delta r_{ij}\, \delta r_{kl}\rangle=\rho_{ij}^2\delta_{ik}\delta_{jl}\ . \tag{3}$$

These fluctuations in the resistance could be produced by an arbitrary noise mechanism. For identical r_{ij}'s, $\rho_{ij}^2=\rho^2$ is assumed to be independent of (ij). Given a constant current source configuration, we calculate the fluctuating part of the resistance R measured in a one-port configuration, the correlation between the measured resistances in a two-port configuration, etc.

To be specific, we denote by

$$\mathscr{S}_R\equiv S_R/R^2=\langle \delta R\, \delta R\rangle/R^2 \tag{4}$$

the relative fluctuation of the measured resistance R (one-port case) due to resistance fluctuations. Similar quantities (see below) will be defined for two-port configurations, etc. S_R so defined is the relevant quantity for the magnitude of $1/f$ noise, measured under constant external current. Other quantities pertaining to other situations can be deduced from S_R through the simple relation

$$\frac{\langle \delta R\, \delta R\rangle}{R^2}=\frac{\langle \delta V \delta V\rangle}{V^2}=\frac{\langle \delta I\, \delta I\rangle}{I^2}\ . \tag{5}$$

As defined by Eq. (4), we need the expression of R as a function of all r_{ij}'s in order to calculate S_R. However, it is clear that such a direct approach leads to formidable calculations and cannot be used in practice. Simplified procedures can be used however: composition rules and sensitivity calculations based on Tellegen's theorem.

A. Composition rules

Starting from the definition of S_R, it is easy to establish the following composition rules for uncorrelated resistance fluctuations.

Series resistances. When two resistances are connected in series, the resulting $\mathscr{S}_R$ for the equivalent total resistance can be written

$$\mathscr{S}_R=(R_1/R)^2\mathscr{S}_{R_1}+(R_2/R)^2\mathscr{S}_{R_2} \tag{6}$$

or more generally,

$$\mathscr{S}_R=\sum_l (R_l/R)^2\mathscr{S}_{R_l}\ , \tag{7}$$

where $R=\sum_l R_l$ is the equivalent total resistance.

Parallel resistances. Similarly, for two resistors in parallel,

$$\mathscr{S}_R=(R/R_1)^2\mathscr{S}_{R_1}+(R/R_2)^2\mathscr{S}_{R_2} \tag{8}$$

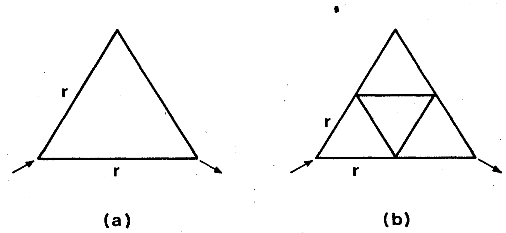

FIG. 1. Two elementary resistor networks where r denotes the common value of the branch resistances. Arrows indicate the location of current probes.

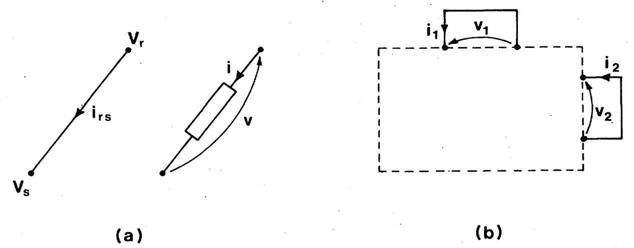

FIG. 2. Sign convention adopted in Tellegen's theorems: (a) voltage across, and current through, a given branch; (b) two-port configuration.

and more generally,

$$\mathscr{S}_R = \sum_l (R/R_l)^2 \mathscr{S}_{R_l} , \tag{9}$$

where $R^{-1} = \sum_l R_l^{-1}$. Here and above, the summation is taken over all branches in the network.

Note the different weights appearing in Eqs. (7) and (9) according to the topology of the network. However, for n identical resistors, arranged either all in series or all in parallel, one obtains the same result: $\mathscr{S}_R \sim 1/n$.

Combined with the star-triangle transformation, the above rules can be used to calculate $\mathscr{S}_R$ for simple networks. For instance, using these simple rules, one obtains, respectively (see Fig. 1), $R = \frac{2}{3}r$ and $\mathscr{S}_R = \rho^2/2r^2$ for the network (a) and $R = 10r/9$, $\mathscr{S}_R = 11\rho^2/50r^2$ for the network (b): Here, R denotes the measured resistance in the corresponding configuration.

B. General theorems

The previous direct approach is clearly not easy to use for large or more complicated networks, like random resistor networks. The more appropriate approach, based on Tellegen's theorems,[17] is called the sensitivity calculation method and is widely used in computer-aided network design. In network design problems, one is generally interested in the following question: What is the amount of change of the potential measured at a port, given that in some branches of the circuit some resistance (or some other circuit element) does not have its prescribed value? This change can be due to some damage or simply to a fluctuation within the prescribed tolerance or to noise. The answer to this question is given in fact by Tellegen's theorems. To be self-contained, we summarize briefly the sensitivity calculation method. For a more detailed discussion, we direct the reader to Ref. 17.

1. Tellegen's theorems

Theorem 1 (simplified version). In a given network, the branch currents i_α and the voltage differences v_α across the corresponding branches satisfy the following sum rule:

$$\sum_\alpha i_\alpha v_\alpha = 0 . \tag{10}$$

The proof of this theorem is somewhat elementary. The set of currents satisfies Kirchhoff's current law (KCL):

$$\sum_{\alpha_n} i_\alpha = 0 . \tag{11}$$

The summation is taken over all branch currents i_α incident upon a given node n. Similarly the voltages v_α satisfy Kirchhoff's voltage law (KVL):

$$\sum_{\alpha_l} v_\alpha = 0 . \tag{12}$$

Here the summation is taken over the branches forming any closed loop l. The power in branch α, given by $p_\alpha = v_\alpha i_\alpha$, can also be written as (see Fig. 2)

$$p_\alpha = i_{rs}(V_r - V_s) = i_{rs} V_r - i_{rs} V_s \tag{13}$$

and taking the sum over all branches, one obtains the claimed result:

$$\sum_\alpha p_\alpha = \sum_\alpha i_\alpha v_\alpha = \sum_r \left[V_r \sum_s i_{rs} \right] = 0 . \tag{14}$$

In Eq. (14) we have used KCL [Eq. (11)].

This general and powerful theorem simply expresses the conservation of energy within the network. The result Eq. (14) holds also for networks possessing different input-output ports (see Fig. 2 for sign convention):

$$\sum_\alpha i_\alpha v_\alpha = \sum_p i_p v_p . \tag{15}$$

In Eq. (15) i_p and v_p correspond to different ports p.

Theorem 2. If a set of voltages v''_α satisfies KVL around all closed loops in a network, and if a set of currents i'_α satisfies KCL at all nodes, then

$$\sum_\alpha i'_\alpha v''_\alpha = 0 . \tag{16}$$

More generally, for a multiple-port network, Eq. (16) can be generalized as above:

$$\sum_\alpha i'_\alpha v''_\alpha = \sum_p i'_p v''_p . \tag{17}$$

This theorem can be proved as was theorem 1. However, the v''_α and i'_α in Eqs. (16) and (17) need not refer to the values at the same instant of time, nor need the branches be the same at the times that the v''_α and i'_α are measured. The only requirements are that the v'_α satisfy KVL, that the i'_α satisfy KCL, and that the associated sign convention be adopted. In addition, nothing needs to be specified about the nature of the network branches. Equations (16) and (17) are a very general theorem. For instance, i'_α can refer to a given network and v''_α to another network having different branch resistances, *but* possessing the same topology.

Theorem 3 (generalized). Let $\mathscr{L}'$ and $\mathscr{L}''$ be two linear operators acting on the v's or on the i's. Within the conditions of theorem 2, we have

$$\sum_\alpha (\mathscr{L}' i_\alpha)(\mathscr{L}'' v_\alpha) = \sum_p (\mathscr{L}' i_p)(\mathscr{L}'' v_p) \tag{18}$$

for an n-port network. In particular,

$$\sum_\alpha [(\mathscr{L}'i_\alpha)(\mathscr{L}''v_\alpha)-(\mathscr{L}''i_\alpha)(\mathscr{L}'v_\alpha)] = \sum_p [(\mathscr{L}'i_p)(\mathscr{L}''v_p)-(\mathscr{L}''i_p)(\mathscr{L}'v_p)] . \quad (19)$$

2. The sensitivity calculation method

Among different applications of the above theorems, two important examples are of interest in the calculation of the noise spectrum S_R.

(i) Cohn's theorem. Assume a one-port configuration for a given network where I denotes the external current and V the voltage between the two contacts. Let R be the resistance $R=V/I$ so measured. Cohn's theorem[18] tells us about the variation of R resulting from a small variation in the value of the resistances inside the network and can be stated as follows:

$$\delta R=\sum_\alpha \delta R_\alpha (i_\alpha/I)^2 . \quad (20)$$

Here δR_α denotes the variation of the value of the resistance in the branch α and i_α the current through this branch in the unperturbed original network.

This basic equation (20) is a consequence of Eq. (19). If $\mathscr{L}'$ is taken to be the identity operator and $\mathscr{L}''$ the small increment δ, then Eq. (19) becomes

$$I\,\delta V-V\,\delta I=\sum_\alpha (i_\alpha\,\delta v_\alpha - v_\alpha\,\delta i_\alpha) , \quad (21)$$

but $v_\alpha=i_\alpha R_\alpha$ and Eq. (20) results immediately.

(ii) Transpose network. Let us consider a multiple-port network N. The voltages (V_a) and currents (I_b) measured at these ports can be related linearly as

$$V_a=\sum_b Z_{ab} I_b , \quad (22)$$

where Z_{ab} are elements of the (square) impedance matrix $\mathbf{Z}$.

The transpose network N' of N is defined as having the same topology as N but with an impedance matrix $\mathbf{Z}'$ which is the transpose of $\mathbf{Z}$: $\mathbf{Z}'=\mathbf{Z}^t$. The application of Eq. (19) to these two networks leads to the following result:[17]

$$\delta R=\sum_{a,b} \delta Z_{ab}(I'_a I_b/I'I) \quad (23)$$

giving the variation of the resistance measured at port 1 resulting from variation of resistances inside the network. The summation is taken over all other ports $a,b=2,3,4,\ldots$ and I' and I refer to the currents [see Eq. (22)] associated with N and N', respectively. Equation (23) holds in particular for two ports and will be used in the next section.

3. Noise calculations

(i) One-port configurations. It is easy to deduce from Eq. (20) the following expression for $\mathscr{S}_R$:

$$\mathscr{S}_R=\langle \delta R\,\delta R\rangle/R^2 = \sum_{\alpha,\beta}\langle \delta R_\alpha\,\delta R_\beta\rangle \left[\frac{i_\alpha}{I}\right]^2 \left[\frac{i_\beta}{I}\right]^2 \Big/ R^2 . \quad (24)$$

However, R can also be written as follows:

$$R=\sum_\alpha R_\alpha (i_\alpha)^2/I^2 , \quad (25)$$

from which we extract

$$\mathscr{S}_R=\sum_{\alpha,\beta}\langle \delta R_\alpha\,\delta R_\beta\rangle (i_\alpha)^2(i_\beta)^2 \Big/ \left[\sum_\alpha R_\alpha (i_\alpha)^2\right]^2 . \quad (26)$$

Only the knowledge of the values of the currents in all the branches of the unperturbed original network and the covariance matrix of the elementary resistance fluctuations are needed to evaluate this general expression for $\mathscr{S}_R$. Furthermore, for uncorrelated fluctuations [Eq. (3)] and identical resistances $R_\alpha=r$, one obtains the simple result

$$\mathscr{S}_R=\frac{\rho^2}{r^2}\left[\sum_\alpha (i_\alpha)^4\right] \Big/ \left[\sum_\alpha (i_\alpha)^2\right]^2 . \quad (27)$$

This expression for $\mathscr{S}_R$ is used extensively in the following sections.

(ii) Two-port configurations. Recently, it has been argued by Weissman *et al.*[16] that precise information about the microscopic origin of $1/f$ noise can be gained by measuring voltage correlation functions using two-port configurations (four-point probe). The notion of the transpose network is very useful for the calculation of such correlation functions (i.e., resistance noise matrix). To see this, let us consider two ports A and B where the measured voltages are V_A and V_B, respectively. According to Eq. (22) we have the following relations:

$$V_A=R_A I_A + R_C I_B,\quad V_B=R_C I_A + R_B I_B , \quad (28)$$

or

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} R_A & R_C \\ R_C & R_B \end{bmatrix} \begin{bmatrix} I_A \\ I_B \end{bmatrix} ,$$

where I_A and I_B are, respectively, the currents at the associated ports. Compare now the two configurations A and B of the same network shown in Fig. 3, where $I_A\neq 0$ and $I_B=0$ in the first, and $I_A=0$ and $I_B\neq 0$ in the second. The difference between configuration A and configuration B is simply the interchange between the probe and the source: The impedance matrix for A is the transpose of that for B and vice versa. Using Tellegen's theorems, one obtains, for the configuration A,

$$\delta V_B=\frac{1}{I_B}\sum_\alpha \delta R_\alpha\, i_\alpha^{(A)} i_\alpha^{(B)} . \quad (29)$$

Here $i_\alpha^{(A)}$ $(i_\alpha^{(B)})$ denotes the branch currents in configuration A (B). Using Eq. (28) one deduces

$$\delta R_C=\delta V_B/I_A=\sum_\alpha \delta R_\alpha \left[\frac{i_\alpha^{(A)}}{I_A}\right]\left[\frac{i_\alpha^{(B)}}{I_B}\right] . \quad (30)$$

Similarly,

$$\delta R_A=\sum_\alpha \delta R_\alpha \left[\frac{i_\alpha^{(A)}}{I_A}\right]^2 , \quad (31)$$

$$\delta R_B=\sum_\alpha \delta R_\alpha \left[\frac{i_\alpha^{(B)}}{I_B}\right]^2 . \quad (32)$$

Equations (30)–(32) can be used to calculate various correlation functions and, in particular, the correlation ratio defined by

$$Q \equiv \langle \delta R_A \, \delta R_B \rangle / \langle \delta R_C \, \delta R_C \rangle \ . \tag{33}$$

In general, Q takes the following form:

$$Q = \frac{\left[\sum_{\alpha,\beta} \langle \delta R_\alpha \, \delta R_\beta \rangle i_\alpha^{(A)} i_\alpha^{(A)} i_\beta^{(B)} i_\beta^{(B)} \right]}{\left[\sum_{\alpha,\beta} \langle \delta R_\alpha \, \delta R_\beta \rangle i_\alpha^{(A)} i_\alpha^{(B)} i_\beta^{(A)} i_\beta^{(B)} \right]} \ . \tag{34}$$

For uncorrelated resistance fluctuations [Eq. (3)], one obtains the following result: $Q=1$. For completely coherent fluctuations and identical resistors, i.e., $R=r$ for all α and $\langle \delta R_\alpha \, \delta R_\beta \rangle = \rho^2$ independent of α and β, one obtains the following results:

$$\mathscr{S}_{R_A} = \mathscr{S}_{R_B} = \mathscr{S}_{R_C} = \mathscr{S}_{R_A,R_B} = \rho^2/r^2 \ , \tag{35}$$

where $\mathscr{S}_{R_A,R_B} = \langle \delta R_A \, \delta R_B \rangle / R_A R_B$. This result agrees with that of Ref. 16, obtained through a two-dimensional continuum approach.

The result $Q=1$ appears as a very general one for uncorrelated resistance fluctuations. $Q=1$ holds for arbitrary networks of any spatial dimension, in particular for the random resistor networks discussed below. Note, however, that it is possible to imagine physical models with correlations between effective resistors representing different directions:[16] For example, a "site" problem, with the conduction done by grains. In such cases, $Q \neq 1$. Note also that for coherent fluctuations [Eq. (35)]

$$Q = R_A R_B / R_C^2 \ . \tag{36}$$

(iii) Higher-order correlation functions. The result Eq. (27) can easily be generalized to higher-order correlation functions. Equation (27) was derived for networks made of identical resistances r which fluctuate independently and have the same ρ. From now on, we restrict ourselves to such a model. Higher-order cumulants of the resistance fluctuations then have a simple expression in terms of the steady-state currents and of the cumulants of the elementary resistance fluctuations (which also are assumed all identical). For example (recall $\langle \delta r \rangle = 0$),

$$\langle (\delta R)^4 \rangle - 3 \langle (\delta R)^2 \rangle^2 = \left[\sum_\alpha (i_\alpha)^8 \right] [\langle (\delta r)^4 \rangle - 3 \langle (\delta r)^2 \rangle^2] \ . \tag{37}$$

We thus define the quantities (n integer)

$$G_{2n} \equiv \sum_\alpha (i_\alpha)^{2n} \tag{38}$$

which for an arbitrary geometry of the network relates microscopic and macroscopic correlation functions of the resistance fluctuations.

Finally, note that in the calculation of the resistance noise correlation functions, only the current patterns in the unperturbed original network $\{i_\alpha\}$ are needed. This situation is reminescent of linear response theory where the response functions (susceptibility for instance) are given by correlation functions pertaining to the unperturbed system.

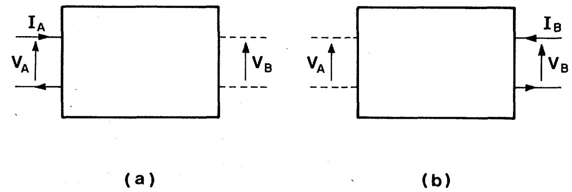

FIG. 3. Network with two ports (four-point probe) used in the calculation of voltage correlation functions.

4. General upper and lower bounds for the noise

Let us consider the expression for the noise given in Eq. (27). It is easy to show that for any current pattern $\{i_\alpha\}$, $N_b \sum_\alpha (i_\alpha)^4 \geq [\sum_\alpha (i_\alpha)^2]^2$ leading to

$$\mathscr{S}_R \geq \frac{\rho^2}{r^2} \frac{1}{N_b} \ , \tag{39}$$

where N_b denotes the number of conducting bonds in the network. In addition, the equality is reached if and only if the currents i_α are all equal. To find an upper bound for the noise, note that the driving current I can be chosen equal to unity which implies that $(i_\alpha)^2 \leq 1$ for all α and hence $\sum_\alpha (i_\alpha)^4 \leq \sum_\alpha (i_\alpha)^2$, from which one obtains

$$\mathscr{S}_R \leq \frac{\rho^2}{r^2} \frac{r}{R} \ . \tag{40}$$

III. NOISE IN SELF-SIMILAR NETWORKS

Before considering self-similar networks, let us consider the case of Euclidean networks made of $L^d d$ resistances arranged in a d-dimensional hypercubic lattice. Assume the noise is measured with two parallel electrodes connecting two opposite faces of the network. The measured resistance is given by $R = r/L^{d-2}$ (L is measured in units of the lattice spacing). Assuming uncorrelated resistance fluctuations, we obtain $S_R = \rho^2 L^{4-3d}$ and

$$\mathscr{S}_R = \langle \delta R \, \delta R \rangle / R^2 = (\rho^2/r^2) L^{-d} \ .$$

This simple result shows that for Euclidean networks $\mathscr{S}_R$ scales as L^{-d}, i.e., the inverse of the volume. A natural question therefore arises: What happens for fractal networks? Is there a new exponent controlling the size dependence of $\mathscr{S}_R$?

This question is motivated by the anomalous size dependence of the resistance $R(L)$ in the case of fractal lattices. More precisely, it has been shown that[3]

$$R(L) \sim L^{-\beta_L} \quad (L \gg 1) \ , \tag{41}$$

where $\beta_L = (\bar{d}/\tilde{d})(\tilde{d}-2)$ is an exponent controlling the transport properties on the considered structure. Here, $\bar{d}$ and $\tilde{d}$ denote, respectively, the fractal and spectral dimensions of the structure.

Another motivation comes from the recent progress in our understanding of random resistor networks. There

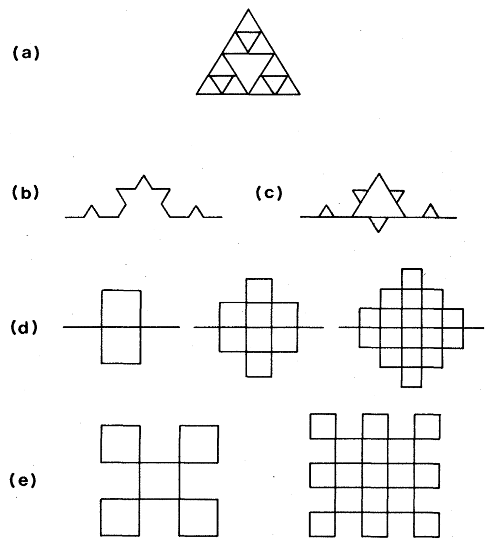

FIG. 4. Fractal lattices for which the resistance noise exponent b is calculated. (a) Stage $n=2$ of the two-dimensional Sierpiński gasket. (b) Stage $n=2$ of the von Koch curve. (c) Stage $n=2$ of the branching von Koch curve. (d) Generators of the "phi lattices" of scaling factor 3, 5, and 7, respectively. (e) Two checkerboard lattices of scaling factor 3 and 5, respectively.

the concept of self-similarity has also been shown to be very useful, particularly in the investigation of the universality[2,3] of the spectral dimension $\tilde{d} \cong 4/3$.

Let us assume that $\mathscr{S}_R$ obeys the following scaling form (this assumption will be supported by specific examples given below):

$$\mathscr{S}_R(\lambda L)=\lambda^{-b}\mathscr{S}_R(L) ,$$

i.e.,

$$\mathscr{S}_R(L)\sim L^{-b} \quad (L \gg 1) , \tag{42}$$

where λ denotes a scaling factor, and b is a characteristic exponent. $\mathscr{S}_R$ is measured in an arbitrary one-port configuration. For Euclidean networks, where $\bar{d}=d$, we have $b=d$. How is the exponent b influenced, if at all, by the geometry?

Before considering specific examples, we give upper and lower bounds for b. On a fractal structure where $N_b \sim L^{\bar{d}}$, we deduce from Eqs. (39) and (42) the upper bound for b: $b \le \bar{d}$. Note that this bound is reached on Euclidean lattices. Using then the general result Eq. (40) and the scaling in Eq. (41) we find another bound, $b \ge -\beta_L$. Combining these two bounds we have

$$-\beta_L \le b \le \bar{d} . \tag{43}$$

Using $\beta_L=(\bar{d}/\tilde{d})(\tilde{d}-2)$, Eq. (43) may be written in the form

$$(2-\tilde{d})/\tilde{d} \le b/\bar{d} \le 1 .$$

One can think of other upper bounds for $\mathscr{S}_R$ (lower bounds for b) but they are not as stringent as the above. For example, $\mathscr{S}_R \le \rho^2/r^2$. Another bound may also be found from the inequality

$$\langle \delta R\, \delta R \rangle \le L_{\min}\rho^2 , \tag{44}$$

where $L_{\min}$ denotes the shortest path between the port nodes. From Eqs. (41) and (44) and[5,6] $L_{\min} \simeq L^{\bar{d}/\hat{d}}$ we obtain

$$-2\beta_L-\bar{d}/\hat{d} \le b . \tag{45}$$

Here $\hat{d}$ refers to the spreading dimension of the structure. This lower bound is not as strict as that in Eq. (43) because $R < L_{\min}$ implies

$$-\beta_L \le \bar{d}/\hat{d} . \tag{46}$$

Note that the latter inequality has not appeared in the literature before.

A. Examples

In order to illustrate these considerations, it is useful to study some fractal structures (see Fig. 4).

(i) 2D Sierpiski gaskets. It is easy to calculate $R^{(n)}$ and $\mathscr{S}_R^{(n)}$ at each stage of the construction ($\beta_L=\ln\frac{3}{5}/\ln 2$, $\bar{d}=\ln 3/\ln 2$)

$$R^{(0)}=\tfrac{2}{3}r,\quad \mathscr{S}_R^{(0)}=\frac{1}{2}\frac{\rho^2}{r^2} ,$$

$$R^{(1)}=\tfrac{10}{9}r,\quad \mathscr{S}_R^{(1)}=\frac{11}{50}\frac{\rho^2}{r^2} ,$$

$$R^{(2)}=\tfrac{50}{27}r,\quad \mathscr{S}_R^{(2)}=\frac{242}{(50)^2}\frac{\rho^2}{r^2} .$$

For this example, Eq. (42) holds trivially ($\lambda=2$) and $\mathscr{S}_R(2L)/\mathscr{S}_R(L)=\frac{11}{25}$. This leads to $b=\ln\frac{25}{11}/\ln 2 =1.1844$ to be compared with $\bar{d}$. As for the exponent β_L, b does not depend on the particular position of the source and drain. For instance, if the current is injected at one corner of the gasket and extracted at the two opposite corners, we obtain the same value of b.

Similarly for the gaskets introduced in Ref. 19 having different scaling factors, we find $b=1.15693$ ($\bar{d}=1.63092$, $-\beta_L=0.6937$) for the scaling factor $\lambda=3$, and $b=1.13112$ ($\bar{d}=1.66096$, $-\beta_L=0.6644$) for $\lambda=4$. Therefore, in this case, increasing $\bar{d}$ results in lowering the value of the exponent b.

(ii) Linear structures. For the von Koch curve, $\bar{d}=\ln 4/\ln 3$, $\beta_L=-\bar{d}$; for the Peano curve, $\bar{d}=2$, $\beta_L=-2$; and for the random-walk trajectory, $\bar{d}=2$, $\beta_L=-2$. For these structures, $\hat{d}=1$ and we obtain $b=\bar{d}$. In these cases, the upper bound of Eq. (43) is reached because of the equality of all branch currents.

(iii) Branching von Koch structure. $\bar{d}=\ln 5/\ln 3$, $-\beta_L=\ln\frac{3}{8}/\ln 3$ and a direct calculation gives $b=\ln\frac{16}{5}/\ln 3$. As for the Sierpiński gasket, due to the presence of loops the exponent b is smaller than $\bar{d}$.

(iv) "Phi lattices." These fractal lattices can be generated for different scaling factors $\lambda=3,5,7,\ldots$. The fractal dimension is the same $\bar{d}=2$ for all λ and $-\beta_L=0.8698$, 0.7415, and 0.6777, respectively, for $\lambda=3$, 5, and 7 (β_L going to zero as $\lambda\to\infty$, as expected). The corresponding values of the exponent b are, respectively, 1.0473 ($\lambda=3$), 1.0044 ($\lambda=5$), and 0.9584 ($\lambda=7$). As λ increases, the exponent b decreases becoming smaller than 1.

(v) "Checkerboards." For the scaling factors $\lambda=3$ and 5 corresponding to the generator shown in Fig. 4, we have, respectively, $\lambda=3$, $\bar{d}=\ln 5/\ln 3$, $\beta_L=-1$ and $\lambda=5$, $\bar{d}=\ln 13/\ln 5$, $\beta_L=-0.8243$. The exponents b we find are, respectively, $b=1$ ($\lambda=3$) and $b=1.3857$ ($\lambda=5$).

Other examples have been studied but we shall limit our discussion only to the above ones. In all cases shown here, Eq. (43) is fulfilled (as it should).

B. A new family of exponents

The new exponent b associated with the noise amplitude $S_R=\mathscr{S}_R R^2$ appears as the third in a series of new exponents associated with the various "moments" of the branch currents. In fact, the first two in this series are given by

$$\sum_\alpha 1 \sim L^{\bar{d}} \quad \text{and} \quad R(L)=r\sum_\alpha (i_\alpha)^2 \sim L^{-\beta_L}$$

as recalled above. The next one is given by S_R which is proportional to $\sum_\alpha (i_\alpha)^4$.

Clearly, all of the above quantities are special cases of the more general G_{2n} defined in Eq. (38). We have seen that the G_{2n}'s have a natural physical meaning even for $n>2$. It is legitimate to associate an exponent x_n to G_{2n} which describes its power-law behavior: $G_{2n}\simeq L^{-x_n}$ for large L. As for b, these exponents can be bounded. In particular $[(i_\alpha)^2 \le 1]$

$$x_n \ge x_{n-1} \tag{47a}$$

for all values of n. Also, the Lyapunov inequality[20] $\mu_n^{1/n} \ge \mu_{n-1}^{1/(n-1)}$ where $\mu_n = G_{2n}/G_0$ implies (for $n>1$)

$$x_n \le x_{n-1}\frac{n}{n-1} + \frac{\bar{d}}{n-1} . \tag{47b}$$

With $x_0=-\bar{d}$, $x_1=\beta_L$, and $x_2=b+2\beta_L$, the last two inequalities immediately lead to Eq. (43). Note also that Eq. (47b) implies $x_n \le n\beta_L+(n-1)\bar{d}$.

The main point we want to emphasize here is that S_R provides a new quantity describing the fractal structure. Quantities like G_{2n} give us further and further details on the fractal structure. The exponents x_n as well as $\hat{d}$ and other exponents introduced for studying the self-avoiding walk statistics[10] appear as exponents containing more and more detailed information about physical properties of fractals.

C. Average resistance and average noise

In general, for a given finite network, there is no natural choice for the contact points in the measurement of R or $\mathscr{S}_R$. In order to avoid this difficulty, it is necessary to define the average resistance and the average noise associated with the considered network.

A natural definition for the average resistance of a finite network is the following:

$$\bar{R} = \sum_{(i,j)} R_{ij} \Big/ \sum_{(i,j)} 1 , \tag{48}$$

where R_{ij} denotes the measured resistance when the current is injected at node i and extracted at node j. The summation is extended over all the possible $N(N-1)/2$ pairs (ij) of the N-node networks. Note that $\bar{R}$ as defined by Eq. (48) is reminiscent of the notion of resistive susceptibility, $\chi_R=\sum_{(i,j)} R_{ij}$, introduced in Refs. 21 and 22.

We define similarly the average noise of a finite network by

$$\bar{\mathscr{S}}_R = \sum_{(i,j)} \mathscr{S}_{ij} \Big/ \sum_{(i,j)} 1 , \tag{49}$$

where $\mathscr{S}_{ij}$ denotes the measured noise using nodes i and j. Similarly, one can define the equivalent of χ_R by $\chi_S=\sum_{(i,j)}\mathscr{S}_{ij}$.

For a fractal network of linear size L, we deduce from the previous discussion the following behavior for $\bar{R}$:

$$\bar{R} \sim L^{-\beta_L}, \quad \chi_R \sim L^{2\bar{d}-\beta_L} . \tag{50}$$

$\bar{\mathscr{S}}_R$ is expected to obey a scaling law:

$$\bar{\mathscr{S}}_R \sim L^{-b} \tag{51}$$

with a characteristic exponent b.

Another quantity of interest would be the average of the resistance fluctuations: $\sum_{(i,j)} \langle \delta R\, \delta R \rangle_{ij} / \sum_{(i,j)} 1$ where $\langle \delta R\, \delta R \rangle_{ij}$ denotes the resistance fluctuations measured between i and j. In general, it should be noted that other averages can be defined. For instance, $\sum_{(i,j)} \langle \delta R\, \delta R \rangle_{ij} / \sum_{(i,j)} (R_{ij})^2$, $\sum_{(i,j)} \langle \delta R\, \delta R \rangle_{ij} / (\sum_{(i,j)} R_{ij})^2$, etc., are other candidates for averaging procedures. However, we believe that Eqs. (48) and (49) are more appropriate for our purpose.

The average resistance as well as the average noise so defined can be calculated for simple systems. For instance on Euclidean lattices, we find $b=d$ as expected. For fractal lattices, we have calculated χ_S for the systems shown in Fig. 5. In Fig. 6 $\chi_S=\sum_{(i,j)}\mathscr{S}_{ij}$ versus the length scale L is shown for (i) branching von Koch structures, $b=1.0587$; (ii) 2D Sierpiński gasket (scaling factor $\lambda=2$), $b=1.1843$; and (iii) the X lattice denoted $C1$ in Ref. 19 (scaling factor $\lambda=2$), $b=1$. The straight lines of slope $L^{2\bar{d}-b}$ show the expected asymptotic behavior (large L) of χ_S. In spite of the small sizes, the slopes reproduce very well the expected values.

IV. APPLICATION TO PERCOLATING SYSTEMS

The fractal geometry of percolation clusters is at the basis of our present understanding of the percolation problem. At $p<p_c$, the random resistor network is divided into isolated finite clusters which are self-similar and have a fractal dimension $\bar{d}_p = d-\beta_p/\nu_p$ and a spectral dimension $\tilde{d}=2(d\nu_p-\beta_p)/(t-\beta_p+2\nu_p)$. At $p>p_c$, the infinite cluster remains self-similar at short length scales up to the correlation length $\xi_p \sim (\Delta p)^{-\nu_p}$, $\Delta p = p-p_c$. The branches carrying the current belong to the backbone. Therefore, the exponent b satisfies

$$-\beta_L \le b \le \bar{d}_B , \tag{52}$$

where $\bar{d}_B$ denotes the fractal dimension of the backbone.

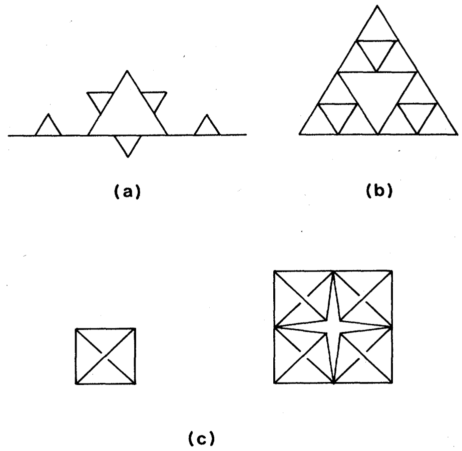

FIG. 5. Fractal lattices used in the calculation of the "noise susceptibility." (a) Branching von Koch structure. (b) Two-dimensional Sierpiński gasket. (c) Two first stages of the construction of the X lattice.

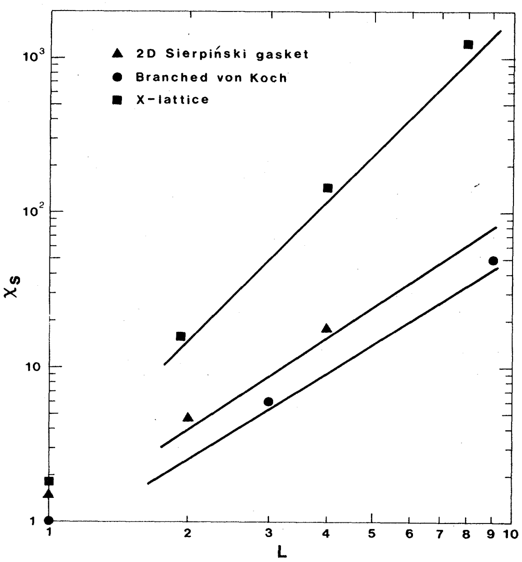

FIG. 6. log-log plot of $[\chi_S]$ for the structures in Fig. 5 [$L \leq 9$ for (a), $L \leq 4$ for (b), and $L \leq 8$ for (c)].

A. Below p_c

The size distribution of finite clusters is given by[23]

$$n(s) \simeq \begin{cases} s^{-\tau}, & s \leq (\xi_p)^{\bar{d}_p} \\ 0, & s \geq (\xi_p)^{\bar{d}_p} \, . \end{cases} \quad (53)$$

Here s denotes the number of bonds inside the finite cluster. The average $[\chi_R]$ of χ_R resulting from the above cluster size statistics is

$$[\chi_R] \simeq \int_0^{(\xi_p)^{\bar{d}_p}} ds\, s^{-\tau} s^{2-\beta_L/\bar{d}_p}$$

$$\sim (\Delta p)^{-\gamma_p + \nu_p \beta_L} \, . \quad (54)$$

In obtaining Eq. (54) we have used the known relation $\tau - 3 = -\gamma_p / \nu_p \bar{d}_p$. This expression can also be written

$$[\chi_R] \sim (\Delta p)^{-\gamma_R} \, , \quad (55)$$

where $\gamma_R = \gamma_p - \nu_p \beta_L = \gamma_p + \zeta$. Here p refers to percolation and R to resistance. Equation (55) is identical to that of Refs. 21 and 22 where it was derived by a different method. In Eq. (55), ζ denotes the exponent $\zeta = t - (d-2)\nu_p = -\nu_p \beta_L$.

The same calculation for the average of χ_s over the cluster size distribution yields

$$[\chi_s] \simeq (\Delta p)^{-\gamma_p + \nu_p b} \simeq (\Delta p)^{-\gamma_s} \, . \quad (56)$$

By analogy with γ_R, we have denoted this exponent by γ_s:

$$\gamma_s = \gamma_p - \nu_p b \, . \quad (57)$$

Using the inequality Eq. (52), we deduce

$$\gamma_p - \zeta \geq \gamma_s \geq \gamma_p - \nu_p \bar{d}_B \, . \quad (58)$$

By using the known estimates of various exponents,[24] we find that γ_s vanishes at $d=1$ and for $d \geq 6$ but takes positive values for $2 \leq d \leq 5$. Therefore, $[\chi_s]$ diverges at finite dimensions as $p \to p_c^-$. The mean-field value ($d \geq 6$) of the exponent b can be obtained from Eq. (52), $-\beta_L \leq b \leq \bar{d}_B$ by taking for the fractal dimension of the backbone[4] $\bar{d}_B = 2$ and for the resistance exponent $\beta_L = -2$. This leads to $b=2$, and $\gamma_s = 0$.

The result for $d \geq 6$ can also be obtained directly by performing a low-p expansion for χ_s at large d. For d large, it is sufficient[21] to keep only self-avoiding walk diagrams. For n-step walks, χ_s is equal to the noise measured between the ends of the walk: $1/n$. Thus

$$[\chi_s] \simeq \sum_n p^n (2d-1)^n \frac{1}{n} \quad (59)$$

which gives a logarithmic divergence at $p_c \simeq 1/2d$. This divergence is consistent with $b=2$ and confirms the above result. Note that at $d \geq 6$ where the mean-field theory is correct, b coincides with its upper bound $\bar{d}_B$ as well as its lower bound $-\beta_L$ [Eq. (52)]. In addition, at $d \geq 6$, all x_n become equal to -2. This value is consistent with the qualitative picture of the percolation clusters viewed as branched polymers[4] at large dimensions.

B. Above p_c

Close to $p=1$, the network is slightly perturbed and its behavior is governed by the Euclidean regime. More precisely, starting from $\mathcal{S}_R = (\rho^2/r^2) L^{-d}$ at $p=1$, $\mathcal{S}_R$ increases when p decreases, satisfying the inequality Eq. (39). The number of conducting bonds N_b decreases, therefore $\mathcal{S}_R$ increases at least as fast as $1/B(p)$. Here $B(p)$ denotes the probability for a given bond to be on the backbone of the infinite cluster. $\mathcal{S}_R$ is therefore expected to follow $1/B(p)$ close to $p=1$. This Euclidean regime is maintained for values of p far from p_c, where $\xi_p \ll L$. For values of p very close to p_c, a Euclidean to fractal

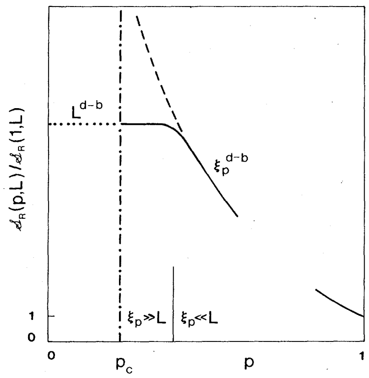

FIG. 7. Variation of $\mathscr{S}_R(p,L)/\mathscr{S}_R(p=1,L)$ as a function of p for a finite box of size L^d. d denotes the Euclidean dimension and b the noise exponent.

crossover is expected to occur.

To be more precise, let us denote by $R(p,L)$ the measured resistance when the potential difference is applied between two opposite $(d-1)$-dimensional faces of a hypercubic box. Similarly, let us denote by $\mathscr{S}(p,L)$ the associated noise. From general arguments, $R(p,L)$ is expected to have the following behavior:

$$R(p,L)=L^{-\beta_L}f(L/\xi_p)\ , \tag{60}$$

where $f(u\ll 1)\simeq 1$, $f(u\gg 1)\simeq u^{2-d+\beta_L}\simeq u^{-t/\nu_p}$. The crossover between the fractal $R(L)\sim L^{-\beta_L}$ and Euclidean $R(L)\simeq L^{2-d}(\Delta p)^{-t}$ takes place at $\xi_p\simeq L$.

Following the same line of argument, the measured noise has its crossover at $\xi_p\simeq L$, between the Euclidean regime $\mathscr{S}_R(p,L)\sim L^{-d}$ and the fractal one $\mathscr{S}_R(p,L)\sim L^{-b}$. Assuming a scaling form for $\mathscr{S}_R(p,L)$

$$\mathscr{S}_R(p,L)=L^{-b}g(L/\xi_p)\ , \tag{61}$$

where $g(u\ll 1)\simeq 1$ and $g(u\gg 1)\simeq u^y$, one deduces $y=b-d$ and $\mathscr{S}_R(p,L)\simeq L^{-d}(\Delta p)^{-\nu_p(d-b)}$ at $\xi_p\ll L$. This behavior is consistent with the increase of $\mathscr{S}_R$ as p approaches p_c from above (recall $b\le\tilde{d}_B\le d$). The expected behavior of $\mathscr{S}_R(p,L)$ is depicted in Fig. 7 where $\mathscr{S}_R(p,L)/\mathscr{S}_R(p=1,L)$ is shown as a function of p. In both regimes , $\xi_p \gtrless L$, the inequalities given by Eq. (43) hold.

V. CONCLUSION

In this paper, we have considered the influence of dilution disorder on the amplitude of $1/f$ noise (or more generally, resistance noise). A new exponent, belonging to a new hierarchy, has been identified first on fractal structures and then on percolating systems. These exponents in a sense characterize the conductance matrix G and can be measured experimentally from higher-order cumulants of the resistance noise. It is not clear whether this family of exponents is related or not to that recently introduced in Ref. 25 to describe the power-law behavior of the resistance cumulants in the random resistor problem. However, from a practical point of view, it is easier to measure the cumulants of the resistance noise[26] than the cumulants of the resistance. It should be pointed out though that in most cases, resistance fluctuations are Gaussian hence higher cumulants do not contain more information and only the exponent b is really meaningful.

The value of this exponent b, which controls the size dependence of the noise, has been calculated in mean-field theory. Both regions $p<p_c$ and $p>p_c$ have been investigated. At $p<p_c$, a new quantity $[\chi_s]$ pertaining to the average noise of finite clusters has been studied. $[\chi_s]$ has been shown to diverge as $p\to p_c^-$ as does the resistive susceptibility $[\chi_R]$. At $p\gtrsim p_c$, an anomalous increase of $\mathscr{S}_R$ has been obtained. Both regimes are controlled by the exponent b. It would be very interesting to calculate b, as well as other exponents by ϵ-expansion techniques[22] at $d=6-\epsilon$, or using numerical simulations.

Finally, two comments are in order. The first is relative to the distinction between the prediction of this model and the diffusion noise model.[27,28] It has been shown[28] that anomalous diffusion near p_c can lead to an additional noise at low frequency, described by $\omega^{1-\nu_{rw}}$ (or $\omega^{1-2\nu_{rw}}$) where $\nu_{rw}=\tilde{d}/2\bar{d}$. In the case discussed in the present paper, no attempt has been made to identify the origin of the frequency dependence. Only the geometrical features have been considered. The second comment is relative to recent measurements of $1/f$ noise in metal-insulator mixtures.[29,30] It would be very interesting to make contact with the predictions given here. In these systems, the change in the amplitude of the noise as a function of filling fraction can indeed be measured.

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