Anomalies in the multifractal analysis of self-similar resistor networks

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Each of the moments of the current distribution in self-similar networks scales with a different exponent. The Legendre transform of these exponents as a function of the order of the moment is called $f(\alpha)$. In general $f(\alpha)$ has a fixed convexity, has a maximum value equal to the usual fractal dimension, is continuous, is positive, and has a finite support $\alpha_{\min}<\alpha<\alpha_{\max}$. Also, it usually characterizes the asymptotic form of the current distribution. Here, explicit examples of physically acceptable exceptions to the behavior of $f(\alpha)$ are exhibited. In the first example, the moments near the zeroth one do not converge uniformly in the large-size limit, leading to an $f(\alpha)$ which has an apparent maximum at a finite value of α while the true maximum is at $\alpha_{\max}=\infty$. In the second example, it is shown that $f(\alpha)$ can take negative values in domains which are relevant for a full characterization of the current distribution. Disorder seems essential to obtain the latter behavior which for these systems compromises the interpretation of $f(\alpha)$ as a continuous set of fractal dimensions.

I. INTRODUCTION

It has been found a few years ago that electrical properties of self-similar resistor networks should be characterized by an infinite set of exponents.[1–4] More precisely, each of the moments of the current distribution is in general controlled by a different exponent. Standard exponents such as the fractal dimension, the resistance exponent and the correlation length exponent (in the case of percolation) are members of this set of exponents, other members of the set being related to comulants of the time fluctuations of the resistance.[1,2]

The fact that an infinite set of exponents is necessary to completely characterize the electrical properties of self-similar resistor networks has analogs in most fields where fractal structures arise, such as turbulence,[5,6] where the concept first appeared,[5] diffusion-limited aggregation,[7] localization,[8] and dynamical systems.[9] What is common to these different fields is that one ascribes a "weight" or "measure" to various parts of a fractal object: To be more specific, one is interested in the current distribution of random resistor networks,[1–4] or in the probability that a site will acquire a neighbor in diffusion-limited aggregation,[7] or in the probability amplitude of a localized eigenstate[8] (or its participation ratio) in localization, or in the close return distances in dynamical systems.[9] The term "multifractal" has been coined[6] to describe the appearance of such infinite sets of exponents. It has also been shown[10] that multifractals are akin to the infinite set of irrelevant exponents which arises in critical phenomena. Much interest in multifractals has been sparked in recent years by the introduction of the Legendre transform[6,11] of the set of exponents, the so-called $f(\alpha)$. This quantity has been interpreted physically as the fractal dimension of subsets whose measure scales as a power α of the length scale.[11]

In the context of resistor networks $f(\alpha)$ has been directly related to the asymptotic shape of the current distribution.[4] In this paper, we first review the $f(\alpha)$ formalism from a slightly different point of view and then we exhibit two simple examples where some properties of $f(\alpha)$ are different from standard expectations. These expectations are that $f(\alpha)$ is concave, has a maximum value equal to the usual fractal dimension, is continuous, is positive, and has a finite support, $\alpha_{\min}<\alpha<\alpha_{\max}$. Only the first two of these properties are rigorously true. The other ones come from observation on numerous systems. In this paper, we exhibit exceptions to the last two properties, positiveness and finite support. In the first example, we find a $f(\alpha)$ which has a value different from the fractal dimension of the backbone at an apparent maximum which occurs at a finite value of α. The true maximum, at which $f(\alpha)$ is equal to the fractal dimension of the backbone, occurs at $\alpha=\infty$. This unusual behavior is due to a nonuniform convergence of the effective exponents describing the scaling properties of the qth moment of the current distribution for q near zero and large system sizes. The physical origin of this nonuniform convergence in turn can be traced to the fact that for this model negative moments do not scale as a power law of the system size. The possibility of nonuniform convergence of the moments near $q=\infty$ has been raised in Ref. 12. However, we show here that nonuniform convergence on intervals of q near the origin is more likely. In the second example, we show that $f(\alpha)$ may be negative, hence, jeopardizing its interpretation as a fractal dimension. While negative $f(\alpha)$ have been seen in other contexts,[13–15] our example clearly traces the origin of this phenomenon to disorder (multifracticality itself can occur even without disorder), and more importantly, it is an example where essential information is lost if the negative part of $f(\alpha)$ is discarded. (This was not the case in the examples studied before Refs. 13–15.)

Our examples are drawn from the general class of hierarchical lattices. Since these lattices are physically realizable,[16] the anomalies of $f(\alpha)$ which we exhibit are not pure mathematical curiosities. Furthermore, it has been widely recognized since the work of Mandelbrot that self-similar objects share many *qualitative* features with physically more relevant statistically self-similar objects such as random resistor networks near the percolation threshold. For example, nonuniform convergence of moments and the corresponding behavior in $f(\alpha)$ which we demonstrate may also occur in percolating systems, for reasons which we discuss in the conclusion.

II. MULTIFRACTAL ANALYSIS

For the sake of completeness, we recall the multifractal analysis of self-similar resistor networks. Let $\mathcal{P}(i^2,L)$ be the fraction of resistors (bonds of the network) which carry a current i in a network of size L. In general $\mathcal{P}$ is a probability distribution. It was found in Refs. 1–4 that the moments of $\mathcal{P}$ behave with size L as a power law,

$$\langle i^{2q}\rangle \equiv \int_0^1 di^2\,\mathcal{P}(i^2,L)i^{2q} \equiv A_q L^{-x_q(L)-D}\ , \tag{1}$$

where D is the fractal dimension of the conducting backbone, and corrections to scaling have been included in the definition of $x_q(L)$. The amplitude A_q is by definition independent of L and is in general a smooth function of q. General results from probability theory[17] then tell us that $-x_q(L)-D$ is a convex decreasing function of q. When the qth moment scales, $x_q(L)$ approaches a limit when L tends to infinity. This generally occurs for positive values of q, but all the negative moments do not necessarily scale, as was demonstrated in the case of percolation by Blumenfeld *et al.*[18] A simple way to account for this is to write the result of Ref. 18 for large L as $x_q(L)=F(L)x_q$.

The asymptotic form of the current distribution $\mathcal{P}(i^2,L)$ can now be derived from the scaling form (1) for the moments: This is the converse of the approach usually taken[6,11,4] where the asymptotic form of $\mathcal{P}(i^2,L)$ is usually assumed and shown to be consistent with Eq. (1). To proceed, it suffices to notice that the *positive* moments $\langle i^{2q}\rangle$ are the Laplace transform of the probability defined by

$$\overline{\mathcal{P}}(-\ln i^2,L)d(-\ln i^2)\equiv\mathcal{P}(i^2,L)di^2\ .$$

Hence, the inverse Laplace transform of the moments may be calculated to obtain

$$\begin{aligned}\overline{\mathcal{P}}(-\ln i^2, L) &= \int_{\epsilon-j\infty}^{\epsilon+j\infty}\frac{dq}{2\pi j}A_q L^{-x_q(L)-D}e^{q(-\ln i^2)}\\ &= L^{-D}\int_{\epsilon-j\infty}^{\epsilon+j\infty}\frac{dq}{2\pi j}\exp\left[-x_q+q\frac{-\ln i^2}{F(L)\ln L}+\frac{\ln A_q}{F(L)\ln L}\right]F(L)\ln L\ ,\quad \epsilon\to 0\ ,\end{aligned} \tag{2}$$

where $j^2=-1$. $-x_q$ is always an analytic function of q for $\mathrm{Re}(q)>0$. On the real axis it is also convex and decreasing. In the limit $\alpha\equiv-\ln(i^2)/[F(L)\ln(L)]$ finite, $L\to\infty$, one can thus use the saddle point approximation to obtain, when A_q depends weakly on q,

$$\overline{\mathcal{P}}(\alpha,L)=C(\alpha,\ln(L))L^{-D}L^{F(L)f(\alpha)}\ , \tag{3a}$$

where $C(\alpha,\ln(L))$ depends weakly on $\ln(L)$ and $f(\alpha)$ is the Legendre transform of x_q, i.e., $\partial x_q/\partial q=\alpha$; $f(\alpha)=q\alpha-x_q$. Note that there are cases[19] where A_q depends strongly on q and, hence, cannot be neglected but this will not be a problem for the examples considered here. Note also that for the saddle-point approximation to apply when $\mathrm{Re}(q)<0$ it suffices that x_q be analytic in that range. The validity of the latter assumption depends on the problem.

In the case where all moments scale, i.e., $F(L)=1$, $f(\alpha)$ has been interpreted,[11] as the fractal dimension of the set of bonds carrying a current which scales with size as $i^2=L^{-\alpha}$. From Eq. (3a), $f(\alpha)$ also suffices to characterize the important features of the probability distribution. As was noted in Ref. 4, however, the moments converge to their asymptotic limit much faster than the distribution reaches its asymptotic form, Eq. (3). A clear example of this is the hierarchical lattice of Refs. 20 and 3, where the exponents reach their asymptotic form at the first level of the hierarchy while even after ten levels (corresponding to 4^{10} bonds) the distribution is still easily distinguishable from its asymptotic form,

$$\lim_{L\to\infty}\frac{\ln\overline{\mathcal{P}}(\alpha,L)}{\ln L}=f(\alpha)-D\ . \tag{3b}$$

The above discussion shows that the asymptotic scaling behavior of the probability distribution can be recovered from the leading scaling behavior of the moments, positive or negative, depending upon where the saddle point is. However, it is worth pointing out that the information contained in the set of all moments is overcomplete. Indeed, since $\mathcal{P}(i^2,L)$ is defined for $0<i^2<1$, one can use theorems from probability theory to reconstruct the current distribution from the *positive integer* moments only.[10] These moments are the experimentally accessible ones in the random resistor networks.[1,2]

III. ACCEPTABLE EXCEPTIONS TO THE BEHAVIOR OF $f(\alpha)$

In the following two examples, we illustrate two exceptions to the $f(\alpha)$ formalism and its interpretation. In Sec. III A, we treat an example where the two limits $L\to\infty$ and $q\to 0$ are not interchangeable, or in other words $x_q(L)$ does not converge uniformly as L tends to infinity for intervals of q which include q equal to 0. In Sec. III B, we show explicitly how disorder may lead to negative $f(\alpha)$ for values of α which cannot be discarded.

A. Nonuniform convergence of x_q in every neighborhood of $q=0$.

The constructions depicted in Fig. 1 lead to a network whose negative moments exist but do not scale and whose behavior around $q=0$ is anomalous. The relevance to percolation is discussed in the conclusion. The lattices in Fig. 1 are constructed by concomitantly iterating the two patterns in Figs. 1(a) and 1(b): At level n, both of the lattices may be considered as a two-port terminal with resistances $R_1^{(n)}$ and $R_2^{(n)}$, respectively; the generation $n+1$ is obtained by connecting $R_1^{(n)}$ and $R_2^{(n)}$ in the same way as $R_1^{(0)}$ and $R_2^{(0)}$ are assembled in cases (a) and (b) of Fig. 1. In this way, given any level n of the hierarchy, both of the networks are made of the same number of resistors, L^D, which defines the fractal dimension as $D=\log_l 7$. Note that from now on, we will use $\log_l$ to denote that the logarithm must be calculated in the base l, where l is the size of the basic pattern which is iterated ($L=l^n$). Whenever a numerical value is needed, such as in the figures of this paper, we use by convention the natural logarithm.

To characterize the scaling properties of this system, let us observe that the ratio of the two resistances $\mathcal{R}^{(n)} \equiv R_1^{(n)}/R_2^{(n)}$ obeys the following recurrence relation:

$$\mathcal{R}^{(n+1)}=\mathcal{R}^{(n)}\left[2+\frac{\mathcal{R}^{(n)}+3}{3\mathcal{R}^{(n)}+1}\right]\left[2+\frac{1+3\mathcal{R}^{(n)}}{3+\mathcal{R}^{(n)}}\right]^{-1} . \quad (4)$$

There is a single attractive fixed point at $\mathcal{R}=1$. Hence, we can restrict ourselves to the neighborhood of this point. It is then convenient to introduce the parameter $1+\epsilon_n \equiv \mathcal{R}^{(n)}=\mathcal{R}_1^{(n)}/\mathcal{R}_2^{(n)}$ whose recurrence relation is, in the limit $\epsilon_n \ll 1$, $\epsilon_{n+1}=2\epsilon_n/3$. Let $\langle i^{2q}\rangle_1^{(n+1)}$ and $\langle i^{2q}\rangle_2^{(n+1)}$ denote the qth moment of the current distribution for the patterns 1 and 2, respectively. Kirchhoff's laws then lead to the following recurrence relation ($\epsilon_0 \ll 1$):

$$\langle i^{2q}\rangle_1^{(n+1)}+\eta\langle i^{2q}\rangle_2^{(n+1)}=\frac{2}{7}\left[1+\frac{\eta}{2^{2q}}\left[1+\frac{3\epsilon_n}{4}\right]^{2q}+\frac{1}{2^{2q}}\left[1+\frac{\epsilon_n}{4}\right]^{2q}+\frac{\epsilon_n^{2q}}{3^{2q}2}\right](\langle i^{2q}\rangle_1^{(n)}+\eta\langle i^{2q}\rangle_2^{(n)}) . \quad (5)$$

where $\eta=\pm 1$. By iterating Eq. (5) with the initial condition $\langle i^{2q}\rangle_1^{(0)}=\langle i^{2q}\rangle_2^{(0)}=1$, the electrical properties of both of the networks are found to be asymptotically identical. The set of effective exponents $-x_q(n)$, defined by Eq. (1) with $A_q=1$, is then given by

$$-x_q(n)=\frac{1}{n}\sum_{k=0}^{n-1}\log_l\left\{2\left[1+\frac{1}{2^{2q}}\left[1+\frac{3\epsilon_k}{4}\right]^{2q}+\frac{1}{2^{2q}}\left[1+\frac{\epsilon_k}{4}\right]^{2q}+\frac{\epsilon_k^{2q}}{3^{2q}\times 2}\right]\right\} . \quad (6)$$

All moments then exist at any finite level of the hierarchy but their scaling properties significantly depend on their sign. For $q>0$, one finds from Eq. (6) that

$$-x_q(\infty)=\log_l 2+\log_l\left[1+\frac{1}{2^{2q-1}}\right], \quad q>0,\ n\gg 1 . \quad (7a)$$

For $q=0$, the sum in Eq. (6) is trivial. One obtains

$$-x_0(\infty)=\log_l 7, \quad q=0 . \quad (7b)$$

Finally, for $q<0$, the series in Eq. (6) diverges in the limit $n\to\infty$. Keeping the dominant term only, one is lead to

$$-x_q(n)\approx nq\log_l(2\epsilon_0/3), \quad q<0 , \quad (7c)$$

which reflects the fact that the smallest current, $(\epsilon_{n-1}\cdots\epsilon_0)/3^n$, in the system dominates all the negative moments. The factor n in Eq. (7c) plays a role analogous to $F(L)$ in Eq. (2). Equation (7) implies that the negative moments do not depend on system size as a power law but that the positive moments do, in close analogy with the case of percolation.[18] From Eqs. (6), (7a), and (7b), one can easily deduce that the limit $q\to 0^+$ and $n\to\infty$ are not interchangeable, and thus the convergence of $x_q(n)$ is not uniform on the whole real positive axis. Physically, when the $q\to 0$ limit is taken last, the "bridge bonds" in Fig. 1 are not counted since symmetrization [see Eq. (4)] leads them to carry a negligible amount of current in the large size limit. On the other hand, if the $q\to 0$ limit is taken before the infinite size limit, all the bonds are counted.

The exponent defined by Eq. (6) is plotted in Figs. 2(a) and 2(b) as a function of q for various system sizes n. On the first coarse scale, Fig. 2(a), $-x_q(n)$ seems to converge to $\log_l 6$ near $q=0$; Fig. 2(b), however, clearly shows the nonuniform convergence on every interval which includes $q=0$. In the infinite size limit, the value at $q=0$ is $\log_l 7$, while the value infinitesimally close to $q=0$ is $\log_l 6$.

Despite the nonuniform convergence, it suffices to consider the $-x_q(n)$ curves for finite values of the size to realize that the Legendre transform is well defined. Hence, the probability distribution for the current at all sizes can be analyzed along the lines of the $f(\alpha)$ formalism, as long as the saddle point lies on the positive definite part of the

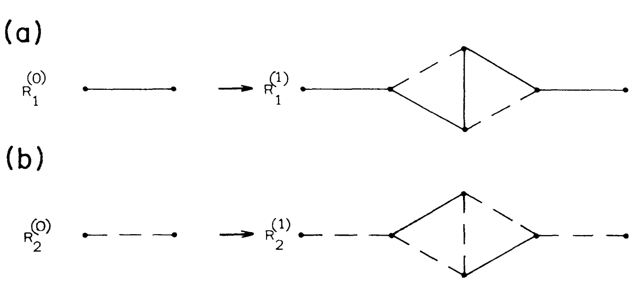

FIG. 1. Patterns which are concomitantly iterated to form a hierarchical structure. Solid lines represent resistance with value $R_1^{(0)}$, and dashed lines, resistances with value $R_2^{(0)}$. The pattern (a) becomes a resistance $R_1^{(1)}$ for the next level of iteration, and the pattern (b) becomes a resistance $R_2^{(1)}$.

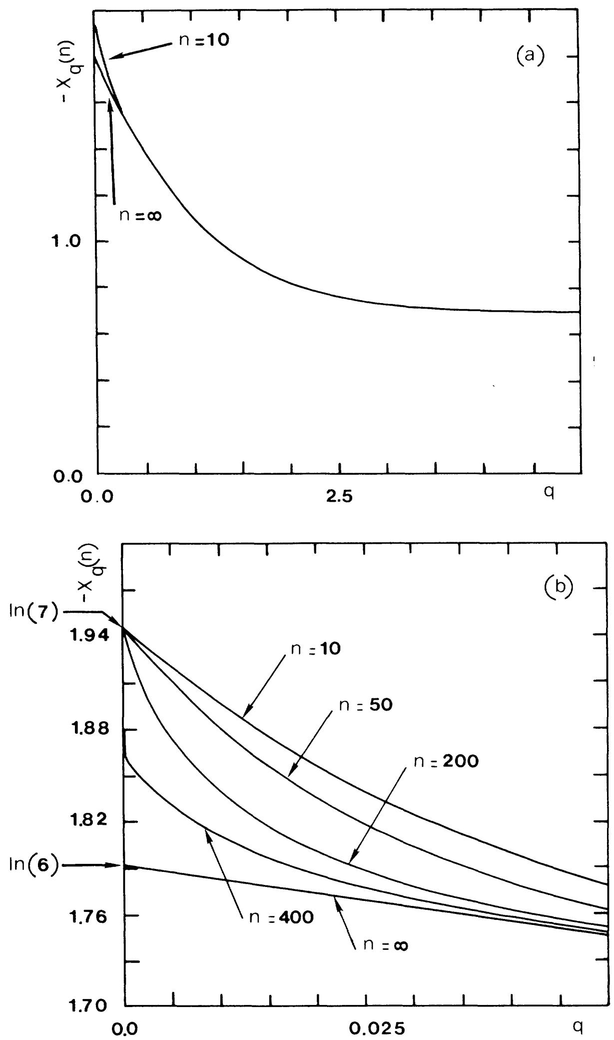

FIG. 2. Exponents, as defined by Eqs. (1) and (6), for the moments of the current distribution in the lattice of Fig. 1. (a) Curve for various values of the level of iteration (coarse q scale). (b) Enlargement of the q scale near the origin. ($\epsilon_0 = 10^{-6}$.)

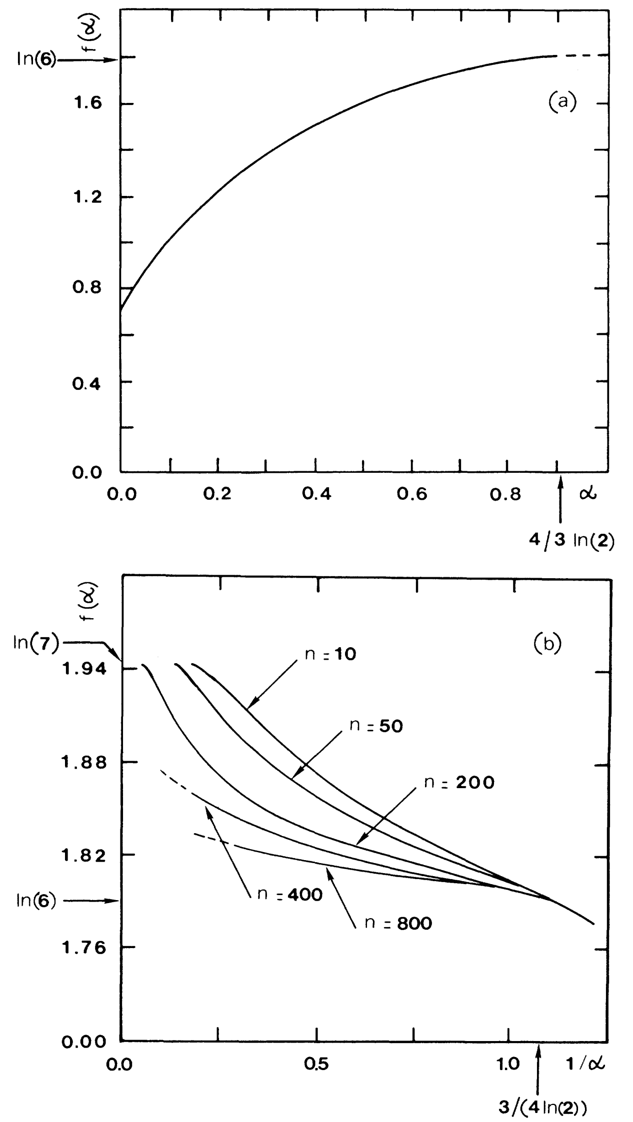

FIG. 3. (a) $f(\alpha)$ curve corresponding to Fig. 2(a). All the curves of Fig. 2(a) coincide on this scale. There is an apparent maximum $f(\alpha) = \ln 6$ at $\alpha = \frac{4}{3}\ln 2$. The region from the apparent maximum to ∞ is represented in part (b) by drawing f as a function of $1/\alpha$. At ∞, f reaches the asymptotic value ln7. ($\epsilon_0 = 10^{-6}$.)

real q axis. Note, however, that $x_q(n)$ can be written here in the form $F(n)x_q$ only for $q > 0$, because of the lack of uniform convergence when the point $q = 0$ is included. The resulting $f(\alpha)$ has an apparent maximum at $\alpha = \frac{4}{3}\log_l 2$ where $f(\alpha)$ takes the value $\log_l 6$, as illustrated in Fig. 3(a). The steep slope near $q = 0$ in Fig. 2(b) leads to ever more positive values of $\partial x_q / \partial q = \alpha$ but to only slight variations of the corresponding $f(\alpha)$. It is more convenient to plot $f(\alpha)$ for $\alpha > (\frac{4}{3})\log_l 2$ as a function of $1/\alpha$. This is done in Fig. 3(b) which shows, for various sizes, how $f(\alpha)$ finally reaches its asymptotic value of $\log_l 7$ at $\alpha = \infty$. The previous discussion clearly shows that one cannot expand the $f(\alpha)$ curve around the maximum; hence, in this example, there is no regime where the log normal is a good approximation.

For $q < 0$, the saddle-point approximation for the current distribution is not applicable since in this region $x_q(n)$ depends linearly on q. Nevertheless, the $f(\alpha)$ curve of Fig. 3 satisfactorily characterizes the whole current distribution. Indeed, the whole probability distribution may be calculated from the positive integer moments[10] and here the amplitudes for these moments are all equal to unity while the corresponding exponents may all be extracted from the $f(\alpha)$ curve by Legendre transformation.

B. Negative $f(\alpha)$

Multifractality and the $f(\alpha)$ formalism are used not only to describe exactly self-similar networks, such as the one just discussed. The same formalism also appears naturally in the context of "random" fractals, such as percolation networks[1-4] and diffusion limited aggregation.[7] In the latter contexts the function $f(\alpha)$ is useful to describe the probability distribution for the "measure" on the fractal. $f(\alpha)$ then has a meaning only when averages over realizations of the disorder are considered. This averaging is absolutely essential since $f(\alpha)$ characterizes properties of "critical" systems, where there is no self-averaging. In this context of scaling disordered systems, $f(\alpha)$ may become negative, something which has not been seen yet in exactly self-similar systems. A simple way to study the effect of disorder consists in constructing a hierarchical lattice whose bonds carry random resistors. This will demonstrate how $f(\alpha)$ may become negative even for values of $\alpha(q)$ corresponding to values of q which are positive and, hence, essential for a complete description of the probability distribution.

Consider resistors drawn from a probability distribution $P(R)\sim R^{-\omega-1}$ (for large R and $0<\omega<1$) and connected in series. Such a problem arises in the calculation of continuum corrections to transport exponents in percolation.[21] One is interested in the distribution of voltage drops in the resistors. Alternatively, one may consider the dual problem of the distribution of currents in conductances drawn from a probability distribution $P(\sigma)\sim\sigma^{\omega-1}$ (for small σ and $0<\omega<1$) and connected in parallel. The model we are considering has some relation to the preceding problem but it also has some important differences. The lattice is built as illustrated in Fig. 4(a). At the first level, pairs of conductances are put in parallel in such a way that the ratio $\gamma<1$ between the two conductances of any given pair has a probability distribution of the form $P(\gamma)\sim\gamma^{\omega-1}$, with $0<\omega<1$. [This can be achieved by drawing the conductances σ from the Lévy stable distributions[22] $P(\sigma)\sim\sigma^{\omega-1}$.] At the next level, pairs of conductances of the previous level are used as basic building blocks. Once again, the ratio between the conductance of two blocks is chosen to be of the form $P(\gamma)\sim\gamma^{\omega-1}$, with γ independent of the previous level. To evaluate the behavior of an arbitrary moment, let us start from the last level, which for definiteness we will assume is the last one in Fig. 4(a). Let $\langle i^{2q}\rangle^{(n+1)}$ be the moment we are looking for, and

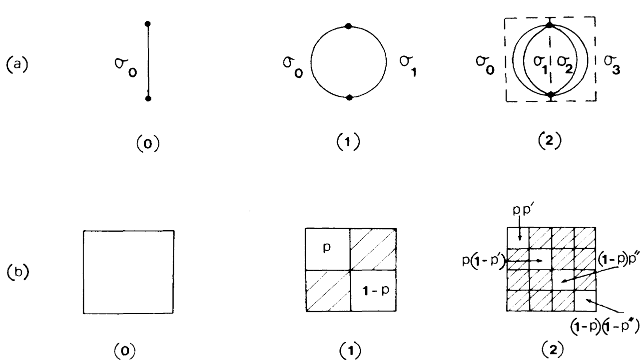

FIG. 4. (a) Construction of a self-similar network of conductances. Each level is built of two conductances from the previous level whose ratio γ is drawn from the probability distribution $P(\gamma)=\omega\gamma^{\omega-1}$. (b) Process of "fragmentation" analogous to the conductance problem. Each of the daughter blocks takes, respectively, a fraction p and $1-p$ of the power dissipated by the mother block [$p=\gamma/(1+\gamma)$ relates the two problems]. The numbers in parentheses refer to the iterations.

$\langle i^{2q}\rangle_l^{(n)}, \langle i^{2q}\rangle_r^{(n)}$ be the moments of, respectively, the left- and right-hand dotted box in Fig. 4(a). We then have the recursion relation,

$$\langle i^{2q}\rangle^{(n+1)}=\frac{1}{2}\left[\left[\frac{\gamma^{(n)}}{1+\gamma^{(n)}}\right]^{2q}\langle i^{2q}\rangle_l^{(n)}+\left[1-\frac{\gamma^{(n)}}{1+\gamma^{(n)}}\right]^{2q}\langle i^{2q}\rangle_r^{(n)}\right], \tag{8}$$

where $\gamma^{(n)}$ is the ratio between the overall conductance of the left- and right-hand box, assuming that the left-hand one has the smallest conductance. $\langle i^{2q}\rangle_l^{(n)}$ and $\langle i^{2q}\rangle_r^{(n)}$ are independent of this ratio since they depend only on γ at a lower level. Denoting the average over disorder by an overbar, one obtains

$$\overline{\langle i^{2q}\rangle^{(n+1)}}=\frac{1}{2}\left[\overline{\left[\frac{\gamma^{(n)}}{1+\gamma^{(n)}}\right]^{2q}}+\overline{\left[1-\frac{\gamma^{(n)}}{1+\gamma^{(n)}}\right]^{2q}}\right]\overline{\langle i^{2q}\rangle^{(n)}}\,. \tag{9}$$

With $\langle i^{2q}\rangle^{(0)}=1$, the above formula may easily be iterated to yield,

$$\overline{\langle i^{2q}\rangle^{(n)}}=\frac{1}{2^n}\left\{\int_0^1 d\gamma\, P(\gamma)\left[\left[\frac{\gamma}{1+\gamma}\right]^{2q}+\left[1-\frac{\gamma}{1+\gamma}\right]^{2q}\right]\right\}^n. \tag{10}$$

Equation (10) may be interpreted in a more transparent way by introducing the variable $p=\gamma/(1+\gamma)$ whose physical meaning can be understood by reference to Fig. 4(b). This is analogous to a process of "fragmentation." The next generation is obtained by dividing the system into two blocks: These daughter blocks dissipate, respectively, a fraction p and $1-p$ of the power in the mother block. Each of these blocks becomes the mother block for the following generation. Our previous analysis is equivalent to considering p as an independent random variable for each generation. Note that in this light, this model is analogous to the random beta model in turbulence.[6,5] The main difference is related to the presence of power conservation in the present model.

From Eq. (10) one deduces that each moment scales with an exponent

$$-x_q=\log_l\left\{\int_0^1 d\gamma\,\omega\gamma^{\omega-1}\left[\left[\frac{\gamma}{1+\gamma}\right]^{2q}+\left[1-\frac{\gamma}{1+\gamma}\right]^{2q}\right]\right\}. \tag{11}$$

A pole first appears with decreasing q at $q=(-\omega)/2$. Nevertheless, one can verify that the saddle-point approximation is valid, but as a consequence of the pole, the function $f(\alpha)$ is defined for α extending all the way to infinity instead of stopping at an $\alpha_{\max}$; also, in the large α limit $\partial f/\partial\alpha$ approaches $-\omega/2$ in contrast to the usual behavior $\partial f/\partial\alpha, -\infty$ as $\alpha\rightarrow\alpha_{\max}$. The Legendre transform of Eq. (11) yields the $f(\alpha)$ curve illustrated in Fig. 5 for the case $\omega=0.5$. One can clearly see that $f(\alpha)$ is positive for a range $\alpha_{\min}<\alpha<\alpha_{\max}$ but negative outside this domain. This means that if we choose to characterize the statistical properties of the dissipated power in terms of subsets of resistors labeled by α, the subsets corresponding to $\alpha<\alpha_{\min}$ or $\alpha>\alpha_{\max}$ have a fractal dimension $f(\alpha)$ less than zero. This usually means that this subset carries a negligible weight.[23] One would then be tempted to restrict $f(\alpha)$ to the finite range of values of α for which $f(\alpha)$ is positive. In such a case, however, the $f(\alpha)$ curve would no longer characterize the probability distribution because an infinite sequence of the positive integer values of $-x_q$ is obtained from the region extending from 0 to $\alpha_{\min}$ which would be dropped in the scenario where negative $f(\alpha)$ are discarded.

The above discussion sets in sharp contrast the average properties of this disordered network with the one obtained when $p=\gamma/(1+\gamma)$ is kept at its median value, $p_m=0.25$ ($\omega=0.5$). In this case, the moments scale with an exponent $-x_q=\log_l[p_m^{2q}+(1-p_m)^{2q}]$ and $f(\alpha)$ is always positive as illustrated in the inset of Fig. 5. As shown in this figure, the positive parts of $f(\alpha)$ in the random and nonrandom cases do not coincide. The above example clearly shows that the physical interpretation of $f(\alpha)$ as a fractal dimension may be open to question when the $f(\alpha)$ formalism is used in the description of scaling properties of disordered systems.[24] On the other hand, we have not found any negative $f(\alpha)$ in the deterministic (nonrandom) case yet. Negative fractal dimensions have an interpretation as "latent" fractal dimensions in the probabilistic context,[25] but they have no counterpart in the deterministic context.

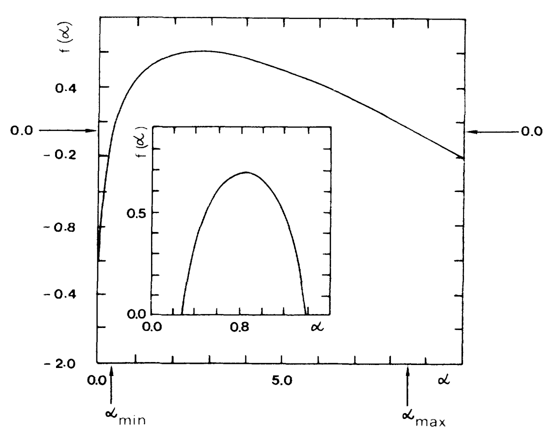

FIG. 5. $f(\alpha)$ curve corresponding to the lattice of Fig. 4, for $\omega=0.5$. Information on the positive moments is contained also in regions of α, where $f(\alpha)$ is negative. Hence, these regions contain essential information. The interpretation of $f(\alpha)$ as a fractal dimension is not so straightforward in regions where it is negative. The inset contains the $f(\alpha)$ curve for the nonrandom model. In the latter case $f(\alpha)$ is always positive.

IV. CONCLUSION

The universal quantities appearing in the description of electrical properties of self-similar resistor networks are the exponents $-x_q$ or equivalently their Legendre transform $f(\alpha)$. Both functions carry exactly the same information. The latter quantity has, however, been given an additional role as a characterization of the current probability distribution[4] as well as a physical interpretation as a fractal dimension.[11] In the present paper, we have studied hierarchical lattices to illustrate two qualitative points: In short, the first example shows that $f(\alpha)$ may be defined on a range of α values which extends on an infinite interval (instead of a finite one) with an apparent maximum at a finite value of α and a true maximum at $\alpha=\infty$. The second example shows that in the context of scaling disordered systems, negative values of $f(\alpha)$ may occur in a range of α which is essential for a complete description of the current distribution. Hence, a complete description of the current distribution requires that $f(\alpha)$ be kept in a regime where its interpretation as a fractal dimension in the usual sense is invalid.[23]

Our two examples are relevant for the study of more realistic systems. Indeed, in our first example, we have shown that the nearly balanced bonds may lead to nonuniform convergence of the exponents $-x_q(L)$ in any interval of q which includes $q=0$. This signals the fact that negative moments do not scale as a power law of network size. A different Lifshitz-type mechanism has been shown,[18] in the context of percolation, to lead to nonscaling negative moments. The divergence of negative moments with size L which we found here is, however, weaker than in Ref. 18. Nevertheless, the results of Ref. 18 strongly suggest that nonuniform convergence of the moments also occurs in percolation for intervals of q which include the negative value of q at which a change in functional dependence of the moments on system size takes place. In a different context, the hydrodynamic dispersion problem, moments of the transit time distribution are analogous to the negative moments,[26] hence, the structure of Fig. 1 may provide another example of a self-similar structure in which the transit time distribution is anomalous, i.e., its moments do not scale as a power law of the system size.[27]

Our second example shows explicitly how disorder analogous to that encountered in the problem of continuum corrections in percolation[21] may lead to moments of order q which do not exist for q sufficiently negative and more importantly may also lead to negative values of $f(\alpha)$. Negative $f(\alpha)$ have been discussed before,[13,15] but in these cases one can always argue that the regions with negative $f(\alpha)$ can be neglected without loss of information since the positive part of $f(\alpha)$ is sufficient, in these cases, to recover the exponents controlling the positive moments (from which the whole distribution may be reconstructed when the amplitudes A_q depend weakly on q). In the present case, the positive moments may not be reconstructed only from the positive part of $f(\alpha)$; hence, regions where $f(\alpha)$ does not have a direct interpretation as a fractal dimension are necessary for a satisfactory characterization of the current distribution. The hierarchical lattice considered in the latter example is somewhat more artificial than that used in the first example, whose positive moments may be made very close to those of percolation. Nevertheless, it is an analog of the random beta model whose properties have been extensively studied in the context of turbulence.[6,5,28] More importantly, it is an example where the ordered version of the lattice has only positive $f(\alpha)$ while the disordered version has negative $f(\alpha)$. This clearly indicates, in the context of disordered systems, that the physical interpretation of $f(\alpha)$ as a fractal dimension in the usual sense fails, unless one extends the interpretation of fractal dimension to negative values. The meaning of negative fractal dimensions has been studied under the name "latent fractal dimensions" by Mandelbrot.[25]

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