Fluctuations in dissipative steady states of thin metallic films

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It is shown that high-frequency current fluctuations induced by the application of a steady electric field to a thin metallic film provide, in certain cases, a way to determine an inelastic-scattering time in a regime where it is otherwise hardly accessible because the usual transport coefficients and equilibrium fluctuations are mainly determined by elastic scattering. The physical meaning of previous calculations on related subjects is also elucidated and extensions of the equilibrium Langevin formalism to nonequilibrium steady states are discussed.

I. INTRODUCTION

The measurement of fluctuations in macroscopic systems probably constitutes the most stringent experimental test of statistical mechanics available. Fluctuations about equilibrium states have been exhaustively studied, thus providing an experimental foundation for the fluctuation-dissipation theorem and for the concept of ensemble, on which equilibrium statistical mechanics rests. This concept, based on the postulate of equal a priori probability, fails in dissipative (equivalently, "nonequilibrium") steady states because external forces may keep the system in states which are highly improbable in equilibrium. There exists, nevertheless, a statistical mechanics of nonequilibrium steady states which reduces to equilibrium statistical mechanics in the appropriate limit. Inasmuch as the few available experiments on fluctuations in nonequilibrium steady states are not always well understood,1 it may be said that nonequilibrium statistical mechanics rests on less solid grounds than its equilibrium counterpart.

A phenomenological statistical basis for the computation of fluctuations in nonequilibrium steady states has been given by Lax.2 He used a Master-equation approach and the general theory of linear Markov processes to describe fluctuations when one-body scattering is the main source of randomization. His results [see, for example, Eq. (2.8) of this paper] were later on generalized and made more rigorous by Van Vliet.3 More recently, many authors (including Van Vliet) have also derived the appropriate analog of Eq. (2.8) for a classical gas with binary collisions. In this case, as well as in the one-body collision case, the collisions may be described as a Poisson process,4,5 a fact which allowed Kogan and Shul'man6 to present a correct set of equations for fluctuations in certain dissipative steady states rather early. Such equations have been derived from the Master equation by many authors, including Ueyama4 and Ernst and Cohen.7 (See Brout8 for a connection between Master equation and Poisson statistics.) Results equivalent to those derived from the Master equation have also been found by diagrammatic methods by Gantsevich et al.9 (see Refs. 10 and 11 for the equivalence between the results of Refs. 6 and 9) and from the Liouville equation by Onuki10 and Ernst and Cohen.7 An important point stressed by Gantsevich et al.12 is that to derive formulas for nonequilibrium fluctuations, no additional hypotheses are necessary beyond those already needed for the validity of the kinetic description. A review of the methods used to describe fluctuations in nonequilibrium steady states, within the kinetic theory of gases, has been given by Ernst and Cohen.7 A review of equivalent methods developed in the context of the electron gas has been written by Gantsevich et al.12

An interesting development has been the recent discovery13 that certain equal-time correlation functions which usually vanish in equilibrium become very slowly decaying functions in a particular nonequilibrium steady state. The formalisms discussed above can be used to do the calculation14 and the results obtained agree qualitatively with experiment.15 By far the simplest method of calculation16 is a local equilibrium extension of the Langevin formalism for hydrodynamic fluctuations about equilibrium. Such a simple extension is, however, not always justified as suggested, among other things, by an earlier calculation17 of current fluctuations in an electron gas scattering off im-
purities in the presence of a constant electric field. There it was shown that the current fluctuations were about 10% larger than what could be expected from simple local equilibrium Langevin arguments. On the other hand, density fluctuations in the same system, but this time subjected to a temperature gradient, have been shown\textsuperscript{18} to agree with expectations from the local equilibrium hypothesis. The latter two calculations were done with many-body techniques which had the advantage of involving very few assumptions but the disadvantage of being very complicated.

In this paper, we first wish to show that the results obtained with the simple method of Lax\textsuperscript{3} agree in detail with the many-body calculations done earlier.\textsuperscript{17,18} We have chosen to do the explicit calculation instead of showing the formal equivalence of the methods because such proofs already exist in other related special cases\textsuperscript{10,11} and because we wish to elucidate the physics of the problems we have previously studied instead of obscuring it further. In particular, we show that, unfortunately, the effect predicted for the current fluctuations of the electron gas in the presence of an electric field is not easily observable in practice. The present paper should also make clear that a local equilibrium description of long-wavelength nonequilibrium fluctuations fails by a few percent when the isotropic part of the one-body distribution function cannot be written in a local equilibrium form. In Sec. II, we present the model and describe the general theory. Section III and the Appendix contain a solution of the above-mentioned problems of fluctuations in an electron gas with one-body impurity scattering.

Section IV contains the main original contributions of this work. There we study a more realistic model of a metallic resistor at low temperatures. We take into account phonons as well as electrons. We make a prediction for high-frequency nonequilibrium current fluctuations which may be used to compare theory and experiment, in cases where the parameters entering the result are known from different sources, or to measure certain parameters if the theory is accepted as valid. In particular, in the latter case, nonequilibrium current fluctuations may provide a way to measure an inelastic-scattering rate in a regime where it is otherwise hardly accessible because the values of the usual transport coefficients and equilibrium fluctuations are dominated by elastic scattering; in nonequilibrium steady states, by contrast with the equilibrium case, the information contained in fluctuations can be quite different from that contained in transport properties. Section V contains remarks on entropy, and Sec. VI concludes with additional physical interpretations and applications of the results of this paper to nonequilibrium superconductivity, refrigeration, and localization theory.

One warning: When we refer to current fluctuations in a metallic resistor, we always refer to fluctuations at frequencies higher than those at which, in real systems, other physical effects tend to give low-frequency (1/f) noise fluctuations.\textsuperscript{1} The frequencies at which current fluctuations are probed should also remain smaller than the high-frequency cutoff imposed by either the external circuitry or the intrinsic relaxation times.

\section{II. SUMMARY OF THE GENERAL THEORY}

The system is described by the following Boltzmann equation:

\[ \frac{\partial f_{\rho\sigma}}{\partial t} + \vec{\nabla}_{\rho} \cdot \vec{V}_{\rho} f_{\rho\sigma} - e \vec{E} \cdot \vec{V}_{\rho} f_{\rho\sigma} = \sum_{\rho',\sigma'} (J_{\rho\sigma,\rho'\sigma'} - J_{\rho',\sigma',\rho} ) \tag{2.1} \]

where \( f_{\rho\sigma}(\vec{r},t) \) is the occupation probability density for a particle of momentum \( \vec{p} \) and spin \( \sigma \) at point \( \vec{r} \) and time \( t \), \( \vec{V}_{\rho} \) is the particle velocity, whose magnitude (to leading order) may be set equal to the Fermi velocity, and \( \vec{E} \) is the electric field. The right-hand side of Eq. (2.1) is the collision operator and

\[ J_{\rho\sigma,\rho'\sigma'} \equiv W_{\rho\sigma,\rho'\sigma'} f_{\rho'\sigma'} (1 - f_{\rho\sigma}) \tag{2.2} \]

\( W_{\rho\sigma,\rho'\sigma'} \) is obtained from the matrix element for transition from state \( \vec{p}'\sigma' \) to state \( \vec{p}\sigma \). We neglect electron-electron collisions. For impurity scattering, we assume that the collisions are elastic, isotropic, and preserve spin, i.e.

\[ W_{\rho\sigma,\rho'\sigma'}^{\text{imp}} = W_{\rho\rho'}^{\text{imp}} \delta_{\sigma\sigma'} = W_{\rho\rho'}^{\text{imp}} \delta_{\sigma\sigma'} \]

\[ = \frac{e}{\hbar} \delta(\epsilon_{p} - \epsilon_{p'}) \delta_{\sigma\sigma'} \tag{2.3} \]

where \( \epsilon_{p} = p^2/(2m) - \mu \) is the chemical potential and \( \delta_{\sigma\sigma'} = 1 \), if \( \sigma = \sigma' \), and zero otherwise. The elastic impurity scattering rate \( e^{-1} \) is then given by

\[ \frac{1}{e} = \sum_{\rho'} W_{\rho\rho'}^{\text{imp}} = N(0) V \int d\epsilon_{p} \delta(\epsilon_{p} - \epsilon_{p'}) w \]

\[ = N(0) V w \tag{2.4} \]

where \( N(0) \) is the single spin density of states at the Fermi surface, and \( V \) is the volume of the system. For scattering by acoustic phonons,\textsuperscript{19}
where \( n(\overline{q}) \) is the occupation probability density for phonons of wave vector \( \overline{q} \), \( M \) is the atomic mass, \( N \) is the number of unit cells in the crystal, \( \omega \) is the phonon frequency, and \( C \) is a constant which is of the order of the Fermi energy \( E_F \): \( C \sim -2E_F/3 \).\(^{19} \) We use a Debye model for the phonon spectrum: \( \omega(q) = \nu_s q \), where \( \nu_s \) is the sound velocity. We do not consider a separate transport equation for the phonons since we will be able to assume that they remain in equilibrium. Note that for time-independent states, the transport theory we use here holds\(^{20} \) even if \( \hbar/\tau_e > k_BT \). \( k_BT \) is Boltzmann’s constant times the temperature.

As a first step, one must find the stationary state by solving Eq. (2.1) with appropriate external fields and/or boundary conditions. This will be done shortly for the special cases of interest. The fluctuations are then found by linearizing the Boltzmann equation around the stationary state \( f_{\text{st}}^{\text{eq}} \). Setting \( f_{\text{st}} = f_{\text{st}}^{\text{eq}} + \delta f_{\text{st}} \) (Ref. 21) in Eq. (2.1), and adding a random Langevin term \( \delta f_{\text{st}} = \sum_{p',\sigma'} (\delta f_{p',\sigma'} - \delta f_{p',\sigma'}^{\text{eq}}) \), one finds for both cases to be considered

\[
\frac{\partial \delta f_{\text{st}}}{{\partial t}} + \mathbf{v}_p \cdot \nabla_p \delta f_{\text{st}} = \sum_{p',\sigma'} \left[ W_{p,\sigma; p',\sigma'} (1 - f_{\text{st}}^{\text{eq}}) - \delta f_{p,\sigma} f_{p',\sigma'}^{\text{st}} \right] \\
- W_{p',\sigma'; p,\sigma} (\delta f_{p',\sigma'}^{\text{eq}} - \delta f_{p',\sigma'} f_{p,\sigma}^{\text{st}}) + \delta f_{p,\sigma} .
\]

(2.6)

We will comment later on the absence of electric field fluctuations in that equation. Kogan and Shul’man\(^6 \) have made the reasonable assumption that the collisions can be described by a Poisson process. Since such a process is built from independent random variables whose mean-square deviation is equal to the mean,\(^22 \) this immediately leads to the following correlation function for the fluctuations:

\[
\langle \delta J_{p_1 \sigma_1 p_2 \sigma_2} (r_1, t_1) \delta J_{p_2 \sigma_2 p_1 \sigma_1} (r_2, t_2) \rangle = V \delta^3(\overline{r_1} - \overline{r_2}) \delta(t_1 - t_2) \delta_{\sigma_1 \sigma_2} \delta_{p_1 p_2} \delta_{t_1 t_2} J_{p_1 \sigma_1 p_2 \sigma_2}^{\text{st}}.
\]

(2.7)

Hence, with \( J_{p_1 \sigma_1 p_2 \sigma_2}^{\text{st}} \equiv J_{p_1 p_2}^{\text{st}} \delta_{\sigma_1 \sigma_2} \) defined as the stationary state value of Eq. (2.2), one finds

\[
\langle \delta J_{p_1 \sigma_1} (r_1, t_1) \delta J_{p_2 \sigma_2} (r_2, t_2) \rangle = V \delta^3(\overline{r_1} - \overline{r_2}) \delta(t_1 - t_2) \delta_{\sigma_1 \sigma_2} \left[ \delta_{p_1 p_2} \left( \sum_{p'} (J_{p_1 \sigma_1 p'}^{\text{st}} + J_{p_2 \sigma_2 p'}^{\text{st}}) \right) - J_{p_1 \sigma_1}^{\text{st}} - J_{p_2 \sigma_2}^{\text{st}} \right].
\]

(2.8)

Note that this correlation function depends on the stationary distribution function. Hence, if the latter deviates greatly from its local equilibrium form, the magnitude of the Langevin-force correlation can be appreciably different from the equilibrium form. Equation (2.6) is a Langevin-Boltzmann equation in which the noise term describes a Poisson process instead of the more usual Gaussian process. It is known that a Langevin description of a Poisson process is possible.\(^23 \)

Equation (2.8) has been derived by a variety of authors\(^8 \) and methods as discussed in the introduction.

### III. FLUCTUATIONS IN A FERMION GAS SCATTERING OFF IMPURITIES

In this section, we neglect the phonons, i.e., we use

\[
W_{p,\sigma; p',\sigma'}^{\text{imp}} = W_{p,\sigma; p',\sigma'} ,
\]

in Eq. (2.2). Two different cases are considered.

A. Density fluctuations in a constant applied temperature gradient

The stationary solution for the distribution function in the presence of a constant temperature gra-
dient is easily found from Eqs. (2.1)–(2.4) by standard methods:

$$f_{p\sigma}^{eq}(\vec{r}) = f_p^{eq}(T(\vec{r})) + \tau_\sigma \vec{\nabla}_p \cdot \vec{\nabla} T \frac{\partial f_p^{eq}(T(\vec{r}))}{\partial \epsilon_p} ,$$  \hspace{1cm} (3.1)

where $f_p^{eq}(T(\vec{r}))$ is a Fermi distribution with a local temperature. To make contact with earlier calculations, we consider a neutral system. Hence, density fluctuations do not generate an electric field and the linearized transport equation can be found using Eqs. (2.6), (3.1), (2.3), and (2.4):

$$\frac{\partial \delta f_{p\sigma}}{\partial t} + \vec{\nabla}_p \cdot \vec{\nabla}_p \delta f_{p\sigma}$$

$$= - \frac{1}{\tau_\sigma} \left( \delta f_{p\sigma} - \int \frac{d\epsilon}{4\pi} \delta f_{p\sigma} \right) + \delta J_{p\sigma} .$$  \hspace{1cm} (3.2)

The moments of that equation will lead to the desired fluctuating hydrodynamic equations.

Summing Eq. (3.2) over spin and momentum gives the continuity equation:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 ,$$  \hspace{1cm} (3.3)

where the number and current densities, $n(\vec{r},t)$ and $\vec{j}(\vec{r},t)$, are defined by

$$n \equiv V^{-1} \sum_{p\sigma} \delta f_{p\sigma} , \hspace{1cm} \vec{j} \equiv V^{-1} \sum_{p\sigma} \vec{\nabla}_p \delta f_{p\sigma} .$$  \hspace{1cm} (3.4)

Note that $\sum_{p\sigma} \delta J_{p\sigma} = 0$. The equation for the current is found by multiplying Eq. (3.2) by $\vec{\nabla}_p$ and then summing over momentum. Neglecting terms of higher order in $l/d$, where $l = v_p \tau_\sigma$ is the mean free path, and $d$ is the characteristic size associated with the gradient, only the first two terms of the expansion of $\delta J_{p\sigma}$ in spherical harmonics are kept. One finds

$$\frac{\partial \vec{j}}{\partial t} + \frac{1}{\tau_\sigma} \vec{\nabla}_p \delta n = - \frac{\vec{j}}{\tau_\sigma} + \vec{\xi} ,$$  \hspace{1cm} (3.5)

where the Langevin force is

$$\vec{\xi} = \tau_\sigma V^{-1} \sum_{p\sigma} \vec{\nabla}_p \delta J_{p\sigma} .$$  \hspace{1cm} (3.6)

The correlation of that random term is easily found by taking moments of Eq. (2.8):

$$\langle \delta j_{p\sigma}(\vec{r}_1, t_1) \delta j_{p\sigma}(\vec{r}_2, t_2) \rangle = V^{-1} \sum_{p\sigma} \left\{ f_p^{eq}(T(\vec{r})) \right\} \left[ f_p^{eq}(T(\vec{r}_2)) + f_p^{eq}(1 - f_p^{eq}) \right] .$$  \hspace{1cm} (3.7)

Neglecting the second order in $\vec{\nabla}_p T$ and performing the angular integrations [see Eqs. (2.3) and (2.4)], the terms proportional to $v_{p1}^l v_{p2}^l$ as well as those coming from the anisotropic part of the distribution function (3.1), disappear and Eq. (3.7) reduces to

$$\langle \delta j_{p\sigma}(\vec{r}_1, t_1) \delta j_{p\sigma}(\vec{r}_2, t_2) \rangle = V^{-1} \sum_{p\sigma} \left\{ f_p^{eq}(T(\vec{r})) \right\} \left[ 1 - f_p^{eq}(T(\vec{r})) \right] \delta(\vec{r}_1 - \vec{r}_2) \delta(t_1 - t_2) ,$$  \hspace{1cm} (3.8)

where $D \equiv \frac{1}{\tau_\sigma} v_p^2 \tau_\sigma$ is the diffusion constant and $\chi_n \equiv 2N(0)$. Given that for time scales larger than the collision time, Eq. (3.5) reduces to

$$\vec{j} = -D \vec{\nabla} n + \vec{\xi} ,$$  \hspace{1cm} (3.9)

and Appendix C of Ref. 16, and they define a simple extension of ordinary fluctuating hydrodynamics.

**B. Current fluctuations in a constant electric field**

We are interested in the current fluctuations to second order in the applied electric field. The first-order correction $f_p^{(1)}$ to the stationary distribution function is, omitting spin indices,

$$f_p^{(1)} = \tau_\sigma e \vec{E} \cdot \vec{\nabla}_p f_p^0 .$$  \hspace{1cm} (3.9)
The second-order correction obeys the equation
\[
\nabla_p \cdot \nabla_{r_p} f_p^{(2)} + \frac{1}{\tau_e} \left[ f_p^{(2)} - \int \frac{dp}{4\pi} f_p \right] = e \bar{E} \cdot \nabla_p f_p^{(1)} . \tag{3.10}
\]

This can be solved to leading order in \(l/d\) by keeping only the two lowest orders of the expansion in spherical harmonics. To leading order in \(k_B T / E_F\), one finds that the momentum-direction-independent part of the distribution function \(f_p^{(2)}\) obeys the equation
\[
\nabla^2 f_p^{(2)} = -(eE)^2 \frac{\partial^2 f_p^{(0)}}{\partial e_p^2} . \tag{3.11}
\]

Even though the solution of that equation will clearly not be a Fermi function with a local temperature, it is instructive to define,
\[
\nabla^2 \delta T = \nabla^2 \left[ \frac{V^{-1}}{C_V} \sum_{p_{\alpha}} e_{p_{\alpha}} \bar{r}_{p_{\alpha}}^{(2)} \right] = -\frac{eE}{k_B} T \frac{3}{\pi^2} , \tag{3.12}
\]

where the constant \(C_V\) is the specific heat of the free-electron gas. We can understand Eq. (3.12) phenomenologically as follows. Since there is no source of inelastic scattering in this problem, a steady state in the presence of an electric field can be reached only if the system also establishes a self-consistent temperature gradient to carry away the Joule heat. Hence, the right-hand side of Eq. (3.12) may be obtained from
\[-\kappa \nabla^2 T_{\parallel} = \left( \mathcal{Y}^2 / R \right) V^{-1} ,
\]

where \(\kappa\) is the thermal conductivity, \(\mathcal{Y}\) the applied voltage, and \(R\) the resistance. Note that there is no density change associated with \(f_p^{(2)}\) in Eq. (3.11) so that our neglect of the self-consistent electric field in finding the stationary solution is justified.

The fluctuations may be found from the Boltzmann equation linearized around the steady state. We consider only uniform current fluctuations in a constant electric field.\(^{24}\) Then, there are no density fluctuations and the linearized Boltzmann equation is given by Eq. (3.2) with \(\nabla \cdot \delta f_p = 0\). Proceeding as in the derivation of Eq. (3.5) one finds, in the limit of noise frequency much smaller than the inverse collision time, \(j = \bar{x}\). The Langevin force correlation is given by Eq. (3.7). As in the previous example, the anisotropic part of \(f_p^{(2)}\) disappears when the angular integrations are performed. There are contributions proportional to \(f_p^{(1)}\) which are treated in the Appendix, but to leading order only \(f_p^{(2)}\) contributes to the nonequilibrium correction to the Langevin-force correlation:
\[
\langle \xi(\bar{r}_1, t_1) \xi(\bar{r}_2, t_2) \rangle_{ne} = 2\tau_e \delta(\bar{r}_1 - \bar{r}_2) \delta(t_1 - t_2) V^{-1} \sum_p \nabla_p^{(1)} \nabla_p^{(2)} (1 - 2f_p^{(0)})
\]

Using Eq. (3.12) in Eq. (3.13), the term in large parentheses is replaced by \((\pi^2 / 9) k_B \delta T\), which confirms the result found in Ref. 17 that the relative increase in current fluctuations is \(\pi^2 / 9\) larger than what would be obtained from the local equilibrium hypothesis [Eq. (3.8)].\(^{25}\) It is clear that this numerical factor comes from the difference between the true stationary distribution function and a Fermi function with a local temperature. The energy-current fluctuations, as another example, are a factor \((\pi^2 / 9)^{1/4}\) larger in this nonequilibrium steady state than in the local equilibrium state. Note also that the corrections to the resistance vanish\(^{26}\) to order \(E^2\); hence there is no simple connection between the nonlinear response and the fluctuations, in contrast with the case of equilibrium fluctuations and linear response.

There are other corrections to the current fluctuations to order \(E^2\) which come, for example, from \(f_p^{(1)} f_p^{(2)}\) in Eq. (3.7) when the first-order correction to both \(f_p^{(1)}\) and \(f_p^{(2)}\) are multiplied. These corrections to the fluctuations are of order \((eE / k_B T)^2\), hence they can be neglected since they are smaller than the result found in Eq. (3.13) by a factor \((l/d)^2\). Nevertheless, some of these corrections were calculated in Ref. 17. They can also be found by the present method and the results agree. Moreover, some of the ambiguities in the interpretation of the results to that order can be clarified by the present method. This will be considered in more details in the Appendix.

The result of Eq. (3.13) could, in principle, be
compared with experiments. We suppose that the measured quantity is the current
\[ \bar{j} = -D\chi_{x} \nabla \mu^{e} + \xi \],

where the double integral is over the sample cross section \( S \). Using
\[ \frac{1}{L} \int_{S} [\mu^{e}(L,y,z,t) - \mu^{e}(0,y,z,t)] = 0 \]  

(3.15)
which is a consequence of the fact that a constant potential difference is applied to the system, one finds from Eqs. (3.12) - (3.14)

\[ S_{f}(\omega) = \int dt e^{i\omega t} (\bar{j}(t)\bar{j}(0))_{ne} \]
\[ = \frac{2}{R} k_{B} \int \frac{d^{3}r}{V} \left[ \pi^{2} \delta T(\vec{r}) \right] , \]

(3.16)

where
\[ R = L/\sigma S = L/(D\chi_{x}e^{2}S) \]
is the resistance and \( \delta T(\vec{r}) \) is the special solution of
\[ -\kappa \nabla^{2} \delta T(\vec{r}) = (\gamma^{2}/R) V^{-1} . \]

The solution of \( \nabla^{2} \delta T(\vec{r}) = 0 \) can be added to account for the boundary conditions but the factor \( \pi^{2}/9 \) should be replaced by unity in these corrections. As an example of the results that Eq. (3.16) predicts, we solved the heat equation for a rectangular sample of dimension \( L_{x}, L_{y}, L_{z} \) with \( L_{z} \ll L_{x}, L_{y} \) and \( \vec{E} \) in the direction \( x \). We find
\[ S_{f}(\omega) = \frac{2k_{B} T}{R} \frac{\pi^{2}}{9} \left[ \gamma^{2} \left( \frac{R_{K} T}{e^{2}} \right) \right]^{2} \]
\[ = \frac{2}{R} k_{B} T \left[ \frac{eE}{6k_{B} T} \right]^{2} , \]

(3.17)

In an experiment to check Eq. (3.16), one must worry about the following complications: (a) In real metals, there is always some inelastic scattering due to the phonons. (b) It is phonons that carry the heat from a metal to an insulating substrate. (c) The Kapitza resistance may cause the temperature jump at the boundary to be much larger than the temperature rise associated with electronic heat conduction; as mentioned below Eq. (3.16), only where \( \mu^{e} \) is the electrochemical potential. Then, the average current in the direction \( x \) along the length \( L \) of the resistor is

\[ I(t) = \int \frac{dx}{L} \int_{S} j(x,y,z,t)dS = \int \frac{dx}{L} \int_{S} \left[ \xi(x,y,z,t) - D\chi_{x} \frac{\partial \mu^{e}(x,y,z,t)}{\partial x} \right] dS , \]

(3.14)

the latter temperature increase must be corrected by the factor \( \pi^{2}/9 \).

To estimate under what conditions inelastic scattering by the phonons can be neglected, we add a phenomenological inelastic scattering rate \( \tau_{p} \) to the right-hand side of the Boltzmann equation:
\[ \frac{\partial f_{p}}{\partial t} + \vec{v}_{p}.\nabla f_{p} - \epsilon \vec{E}.\nabla f_{p} = \frac{1}{\tau_{p}} \int \frac{d^{3}p}{4\pi^{2}} f_{p} - \frac{1}{\tau_{in}} f_{p} . \]

(3.18)

In the limit \( \tau_{e} \ll \tau_{in} \), this leads to the following equation for the isotropic part of \( f^{(2)} \):
\[ \begin{bmatrix} D \nabla^{2} + \frac{1}{\tau_{in}} \end{bmatrix} f_{p} = \frac{1}{3} \left( \frac{eE}{\tau_{e} \delta^{2}} \frac{\partial^{2} f^{(2)}}{\partial p^{2}} \right) . \]

(3.19)

Hence, it is necessary to have
\[ d \ll \left[ \frac{\tau_{in}}{\tau_{e}} \right]^{1/2} \]

(3.20)
(where \( d \) is the characteristic size of the temperature gradient in the system), if we want \( f^{(2)} \) to be given by Eq. (3.11) and the fluctuations by Eq. (3.16).

Furthermore, because the phonons carry the heat from the metal to the insulating substrate [complication (b) above], the distribution function close to the surface will not be given by Eq. (3.11) in a layer of thickness of the order of the phonon mean free path \( l_{ph} \). Hence, the result of Eq. (3.16) will apply to the system only if
\[ d \gg l_{ph} . \]

(3.21)

Finally [complication (c)], the temperature jump due to the Kapitza resistance \( R_{K} \),
\[ \frac{\Delta T}{T} = \frac{(\sigma E^{2}) R_{K} d}{T} , \]

(3.22)
must be much less than the temperature rise given by Eq. (3.12). This leads to
\[ d >> 12 R_k T \sigma \left[ \frac{k_B}{e} \right]^2 \frac{\pi^2}{3}. \] (3.23)

Hence, it will be possible to check Eq. (3.16) experimentally only in systems where Eqs. (3.20), (3.21), and (3.23) can be simultaneously satisfied. We estimate the parameters entering these inequalities as follows. The inelastic scattering time will be evaluated in the next section; for the moment, we take \( \tau_m \approx \tau_0 \), where

\[ \frac{1}{\tau_0} \equiv 3 \pi \left[ \frac{m}{M} \right] \frac{C^2}{\hbar^2 k_F v_s} \left[ \frac{T}{\Theta_D} \right]^{1/3} \zeta(3). \] (3.24)

\( m \) is the electron effective mass, "zeta" is Riemann's "zeta" function and \( \Theta_D \) is the Debye temperature

\[ (k_B \Theta_D)^3 = 6 \pi^2 n (\hbar \nu_s)^3, \] (3.25)

where \( n \) is the inverse of the volume per unit cell. From now on, we assume that we have a monovalent metal so that

\[ n = \frac{k_F^3}{3 \pi^2}. \] (3.26)

For the phonon mean free path, we use the formula which is appropriate for \( q l << 1 \), where \( q \) is the phonon wave vector. This is imposed by the fact that one has to work at temperatures of the order of 1 K and with relatively dirty films for the effect to be observable. Hence,\(^{28}\)

\[ l_{ph} = \frac{15}{8} \frac{M v_s^2}{v_n E_F \tau_e \hbar \omega^2}. \] (3.27)

Using Eqs. (3.24) – (3.27) and the Bohm-Staver relation for monovalent metals,

\[ v_s^2 = \frac{m}{3M} v_F^2, \] (3.28)

one finds that Eqs. (3.20) and (3.21) can be simultaneously satisfied only if

\[ 5.16 \left[ \frac{\hbar}{\tau_e} \right]^3 \frac{(k_B T)^3}{(\hbar \omega)^4 E_F^2} << 1. \] (3.29)

If we take typical phonon frequencies to be of the order of \( k_B T / \hbar \), then at \( T \sim 1 \) K and for \( E_F \sim 7 \) eV, we find

\[ \tau_e >> 10^{-14} \text{sec}. \] (3.30)

If it is phonons of a lower frequency which determine the mean free path, as suggested by Perrin,\(^{27}\) then at worst we should use \( \hbar \omega \sim k_B T / 50 \), in which case

\[ \tau_e >> 10^{-12} \text{sec}. \] (3.31)

Both Eqs. (3.30) and (3.31) are consistent with \( \tau_e \ll \tau_0 \).

To find under which conditions Eqs. (3.23) and (3.20) are compatible we use,

\[ R_k \sim \alpha \frac{30 \hbar^2 v_s^2}{\pi^3 (k_B T)^3 k_B}. \] (3.32)

When \( \alpha = 1 \), we recover the "black-body" formula which provides a sort of lower bound on \( R_k \). With this formula, we find that Eqs. (3.20) and (3.23) are consistent only if

\[ \frac{3.45 \times 10^{-3}}{\alpha^2} \left[ \frac{\hbar}{\tau_e} \right] \left[ \frac{M}{m} \right] \left[ \frac{k_B T}{E_F} \right]^{1/3} \frac{1}{k_B T} >> 1. \] (3.33)

An experiment is realizable only if (3.33) and (3.29) can be simultaneously satisfied. Eliminating \( \tau_e \) from these inequalities, the only condition left is

\[ 500 \alpha^2 \left[ \frac{M}{m} \right] \left[ \frac{E_F}{\hbar \omega} \right]^{4/3} \ll 1. \] (3.34)

In the best case, we can take \( \hbar \omega \sim k_B T \sim 10^{-3} \) eV, \( E_F \sim 1 \) eV, \( m / M \sim 10^{-4} \), and \( \alpha = 1 \). Even with these numbers, the inequality (3.34) cannot be satisfied. Note, however, that if \( \alpha \sim 10^{-1}, 10^{-2} \), then the experiment becomes realizable. Similar conclusions are reached when the formula appropriate to \( q l >> 1 \) is used for \( l_{ph} \) instead of Eq. (3.27).

IV. ELECTRONS SCATTERING OFF IMPURITIES AND PHONONS

In this section, we consider the case of a metallic film of thickness \( d \) satisfying,

\[ d << l_{ph}. \] (4.1)

We will verify at the end that under certain circumstances, it is self-consistent to assume that the phonons are in good thermal contact with the reservoir and that the electron distribution function is uniform. Hence, we take a Bose distribution for the phonon distribution function while the electron distribution function is found by solving the Boltzmann Eqs. (2.1) and (2.2) with

\[ W_{po,p' o'} = W_{po,p' o'}^{imp} + W_{po,p' o'}^{ph}. \] (4.2)

As before, we assume that \( \tau_e \) is much less than the
inelastic scattering time and we expand the distribution function in powers of the electric field: 

\( f = f^0 + f^{(1)} + f^{(2)} \). It is only \( f^{(2)} \) that contributes to the corrections to the current fluctuations because \( f^{(1)} \) is still given by Eq. (3.9) and has a vanishing angular average. The second-order contribution to the Boltzmann equation is, defining \( \overline{\mathbf{p}}' = \overline{\mathbf{p}} + \overline{\mathbf{q}} \),

\[
- \tau(e\mathbf{E} \cdot \mathbf{v}_p) \frac{\partial^2 f^0_p}{\partial \epsilon_p^2} = \frac{\pi C^2}{M} \frac{1}{N} \int \frac{d^3 q}{(2\pi)^3} \frac{q^2}{v_s q} \left\{ [f_p^{(2)}(1+n_q - f_p^0) - f_p^{(2)}(n_q + f_p^0)] \delta(\epsilon_p - \epsilon_p' + \mathbf{f}_s q) \\
+ [f_p^{(2)}(n_q + f_p^0) - f_p^{(2)}(1+n_q - f_p^0) + f_p^{(1)} f_p^{(1)}] \delta(\epsilon_p - \epsilon_p' - \mathbf{f}_s q) \right\} 
\]

\[
- \frac{1}{\tau} \left[ f_p^{(2)} - \int \frac{d\hat{\phi}}{4\pi} f_p^{(2)} \right].
\]  

(4.3)

We first note that the term proportional to \( f_p^{(1)} f_p^{(1)} \), which acts like a source term for \( f^{(2)} \), is negligible compared with the left-hand side of Eq. (4.3). Indeed, to leading order, these terms may be written

\[
P(f^{(1)} f^{(1)}) = \frac{1}{\tau_0} \frac{1}{7\xi(3)} \int_0^\infty dx x^2 [f_a^{(1)}(\xi - x) - f_a^{(1)}(\xi + x)] f_a^{(1)}(\xi) (\hat{\mathbf{p}} \cdot \mathbf{E})^2,
\]

(4.4)

where \( \tau_0 \) is defined in Eq. (3.24) and

\[
\xi = \frac{\epsilon_p}{k_B T},
\]

(4.5a)

\[
x = \frac{\hbar \omega}{k_B T},
\]

(4.5b)

and

\[
f_p^{(1)} = f_a^{(1)}(\epsilon_p) \hat{\mathbf{p}} \cdot \mathbf{E}.
\]

(4.5c)

Defining

\[
| I \rangle = - \tau(e\mathbf{E} \cdot \mathbf{v}_p) \frac{\partial^2 f^0_p}{\partial \epsilon_p^2},
\]

(4.6)

we find,

\[
\frac{P(f^{(1)} f^{(1)})}{| I \rangle} = \tau - \frac{1}{7\xi(3)} \int_0^\infty dx x^2 \left[ \frac{\partial f^0(\xi - x)}{\partial \xi} - \frac{\partial f^0(\xi + x)}{\partial \xi} \right] \frac{\partial f^0}{\partial \xi} \frac{\partial f^0}{\partial \xi^2},
\]

(4.7)

which is much less than unity since \( \tau_e < \tau_0 \), while the integral times the various derivatives of Fermi functions appearing in Eq. (4.7) is finite for the relevant range of values of \( |\xi| \leq 1 \).

To solve the remaining integral equation (4.3), we first notice that since elastic scattering relaxes all contributions to \( f^{(2)} \), except that which is independent of the direction of momentum, the latter will be the leading contribution. It will be proportional to \( \tau_0 \) instead of \( \tau_e \). Hence, it suffices to solve the equation which is obtained by averaging Eq. (4.3) over angles. Denoting the angular average of \( | I \rangle \) and \( f^{(2)} \) by \( | \overline{F} \rangle \) and \( \overline{f^{(2)}} \), and reintroducing the time dependence, the average of Eq. (4.3) may be written symbolically in the form,

\[
\frac{\partial}{\partial t} | \overline{F}^{(2)} \rangle + | \overline{F} \rangle = K | \overline{F}^{(2)} \rangle,
\]

(4.8)

where \( K \) is the collision operator. To treat the scattering-in term of the collision operator, we resort to the mutilated collision operator method of Eckern and Schöhn. Defining the scalar product

\[
\langle f | g \rangle = \int_{-\infty}^\infty d\xi f(\xi) \cosh^2 \left\{ \frac{\xi}{2} \right\} g(\xi),
\]

(4.9)

it can then be verified that \( K \) is Hermitian and has only negative eigenvalues and one zero eigenvalue.
due to the conservation of particle number. The corresponding eigenvector is

\[ |N\rangle = \cosh^{-2}(\xi/2), \tag{4.10} \]

which can easily be found from the change in the equilibrium distribution due to a chemical potential shift. Following Ref. 29, it is then useful to define a reduced collision operator which has the above properties of the true collision operator but which

\[ \frac{1}{\tau_{\text{in}}(\xi)} = \frac{1}{\tau_{0}} \frac{1}{7\xi(3)} \int_{0}^{x_m} dx x^2 [n(x) + f^0(\xi + x) - n(-x) - f^0(\xi - x)], \tag{4.11} \]

where \( x_m = \hbar \omega_D/k_B T \), with \( \omega_D \) representing the Debye frequency. Using the reduced collision operator \( K_{\tau} \), the Boltzmann equation (4.8) is easy to solve. Using \( \langle N | \tilde{T} \rangle = 0 \) and \( \langle N | K_{\tau} | f^{(2)} \rangle = 0 \), one first finds \( \langle N | f^{(2)} \rangle = 0 \), and then

\[ \frac{1}{\tau_{\text{in}}(\xi)} = \frac{1}{\tau_{0}} \frac{1}{7\xi(3)} \int_{0}^{x_m} dx x^2 [n(x) + f^0(\xi + x) - n(-x) - f^0(\xi - x)], \tag{4.12} \]

From Eqs. (4.10), (4.12), and (4.6), \( \tau_{\text{in}} \) and \( \langle N \rangle \) are even in \( \xi \), and \( |\tilde{T}\rangle \) is odd; hence \( \langle N | \tau_{\text{in}} | \tilde{T} \rangle = 0 \), and Eq. (4.13) reduces to

\[ f^{(2)} = -\tau_{\text{in}}(\xi) |\tilde{T}\rangle \]

and may also be written

\[ f^{(2)} = 3\tau_{\text{in}}(\xi) 2 f^0 \frac{eE_F}{k_B T} \frac{\partial^2 f^0}{\partial T^2}. \tag{4.14} \]

The inelastic collision time defined in Eq. (4.12) may be evaluated analytically for small values of \( \xi \):

\[ \lim_{\xi \to 0} \frac{1}{\tau_{\text{in}}(\xi)} = \frac{1}{\tau_{0}} \left[ 1 + \frac{2 \ln 2}{7\xi(3)} \xi^2 + \cdots \right]. \tag{4.15} \]

To compute the current fluctuations to better than 2\%, one needs \( \tau_{\text{in}}(\xi) \) for \( \xi \) in the range \(-4.5 \leq \xi \leq 4.5\). We computed the integral numerically in that range and found that the formula

\[ \tau_{\text{in}}^{-1}(\xi) = \tau_{0}^{-1} 1.34(cosh(\xi/2) - 0.255) \tag{4.16} \]

fits the numerical results to better than 3\%.

In Fig. 1, we compare the distribution function obtained from Eqs. (4.14) and (4.16) with the local equilibrium result

\[ f^{(2)}_{\text{eq}} = \frac{\partial f^0}{\partial T} \delta T, \]

where \( \delta T \), the effective temperature rise, is determined from energy considerations, is simpler to manipulate:

\[ K_{\tau} |f\rangle = -\tau_{\text{in}}^{-1} |f\rangle \]

\[ + \tau_{\text{in}}^{-1} \langle N | \tau_{\text{in}}^{-1} |f\rangle \langle N | \tau_{\text{in}}^{-1} |N\rangle \quad \tag{4.17} \]

where the first term on the right-hand side is identical to the inelastic scattering-out rate of the true collision operator Eq. (4.3):

\[ 8T = 2N(0)C_{\tau}^{-1} \int_{-\infty}^{\infty} \frac{d\epsilon_p e_{p}}{\epsilon_{p}} \frac{\partial^2 f^0}{\partial T^2} \frac{\partial f^0}{\partial T^2} / (k_B^2 T^2). \tag{4.17} \]

Clearly, the true perturbed distribution function is quantitatively very different from what local equilibrium would predict, even though the overall qualitative shapes are comparable.

The corrections to the equilibrium current fluctuations may be computed from Eq. (2.8). Since \( \tau_{e} \ll \tau_{\text{in}} \), in other words since the resistance is determined mainly by elastic scattering, we find that to leading order we can use Eq. (3.7). Hence,

\[ f^{(2)}_{\text{eq}} = \frac{\partial f^0}{\partial T} \delta T, \]

where \( \delta T \), the effective temperature rise, is determined from energy considerations.
The evaluation of that quantity involves the integral
\[
\int_{-\infty}^{\infty} dy \frac{1}{\sinh y} \frac{1}{\cosh y - 0.26} ,
\]
which may be evaluated to better than 3\% by keeping three terms in the expansion of the denominator in powers of 0.26/cosy. Hence, one obtains from Eq. (4.18) the nonequilibrium correction to the current fluctuations, in the limit of \( \omega < \tau_e^{-1} \),
\[
S_I(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle I(t) I(0) \rangle_{ne} = \frac{2}{R} k_B T \left[ \frac{eE \tau^*}{k_B T} \right]^{2/3} ,
\]
where \( \tau^* = v_0^2 \tau_0 \tau_e \) and \( R \) is the resistance as determined by the elastic scattering time only. With the use of \( \delta T / T \) calculated from Eq. (4.17), the above result may also be written
\[
S_I(\omega) = \frac{2}{R} k_B (1.27 \delta T) .
\]
This last way of writing the result elucidates the physical origin of the nonequilibrium current fluctuations, but by contrast with the problem considered in the preceding section, there is no reliable phenomenological way of determining the temperature rise \( \delta T \). The factor 1.27 in Eq. (4.21) becomes 5.33 when the effective temperature rise is evaluated from \( C_p \delta T / \tau_0 = (v^2 / R) V^{-1} \) instead of from Eq. (4.17).

Our result (4.20) can be quite useful since it shows that nonequilibrium current fluctuations are a way to measure the inelastic mean free path \( \tau^* = v_0^2 \tau_0 \tau_e \) in a regime where it is usually inaccessible because elastic scattering dominates the contributions to the value of the usual transport coefficients or equilibrium fluctuations. The conditions under which Eq. (4.20) applies are also quite realistic. Comparing the heating of the phonons due to the Kapitza resistance, Eq. (3.22), with the effective temperature rise of the electron distribution, one finds that the latter is much larger than the former if
\[
d << 1.15 \left[ \frac{m}{M} \right] \left[ \frac{E_F}{m} \right]^{1/2} \left[ \frac{\hbar}{k_B T} \right]^{4/3} \frac{1}{k_B R_k} .
\]

Using \( E_F \sim 7 \text{ eV}, \ M / m \sim 6 \times 10^4, \ m \sim 9 \times 10^{-28} \text{ g}, \) and \( R_k T^3 \sim 10^{-10} \text{ cm}^2 \text{K}^3 / \text{sec} / \text{erg} \), which is a realistic value for the Kapitza resistance, one finds from Eq. (4.22),
\[
d << \frac{6 \times 10^{-4} \text{cm}}{T} .
\]

Thus, for films of thickness of the order of 1000 Å and temperatures of the order of 1 K, phonon heating is negligible. This result is consistent with the fact that the inelastic collision time of electrons with phonons [see Eqs. (3.24)–(3.26) and Eq. (3.28)],
\[
\tau_0 = \frac{1}{9.91} \left[ \frac{m}{M} \right] \left[ \frac{E_F}{k_B T} \right]^2 \frac{\hbar}{k_B T} ,
\]
is of the order of \( 10^{-7} \) sec. at 1 K, while it typically takes \( 10^{-10} \) sec. for a phonon to cross a \( 10^{-5} \)-cm thick film. For thicknesses of that order, the condition (4.1) can also be satisfied. For any particular experiment Eq. (3.27) may be used with \( \hbar \omega \sim k_B T \) to check Eq. (4.1).

Finally, it should be pointed out that Eq. (4.20) leads to sizeable effects in relatively small fields. For example, at \( T \sim 1 \text{ K} \) with \( \tau_e \sim 10^{-14} \text{ sec} \) and the other numerical values quoted above, one obtains a 10\% effect in a field of the order of \( 10^{-2} \text{ V/cm} \).

**V. REMARKS ON ENTROPY**

It is well known that equal-time fluctuations about equilibrium may be computed from entropy considerations only. In nonequilibrium steady states, this is not in general possible, as pointed out for example by Lax\(^2\) and Gantsevitch \textit{et al.}\(^1\) To show this explicitly, the latter derived a natural extension of the equilibrium formula for entropy by using the usual combinatorial argument for fermions with the additional constraint that on the average, the occupation of each state is given by the stationary solution of the Boltzmann equation. They obtained
\[
S = -k_B \sum_{\rho \sigma} f_{\rho \sigma} \ln \left[ \frac{f_{\rho \sigma}}{f_{\rho \sigma}^{eq}} \right]
\]
\[
+ (1 - f_{\rho \sigma}) \ln \left[ \frac{1 - f_{\rho \sigma}}{1 - f_{\rho \sigma}^{eq}} \right] .
\]

Defining \( f_{\rho \sigma}^{eq} = f_{\rho \sigma}^{eq} + \delta f_{\rho \sigma} \), and using the Einstein formula \( P = \exp[S / k_B] \) for the probability distribution of fluctuations \( P \), one finds
\[
\langle \delta f_{p_1 \sigma_1} (\vec{r}_1, t) \delta f_{p_2 \sigma_2} (\vec{r}_2, t) \rangle
\]
\[
= \nabla \delta_{p_1 \sigma_1} \delta_{\sigma_2 p_2} \delta^2 (\vec{r}_1 - \vec{r}_2) f_{p_1}^{\text{eq}} (\vec{r}_1) \times [1 - f_{p_1}^{\text{eq}} (\vec{r}_1)] .
\] (5.2)

For fluctuations about equilibrium, \( f_{p\sigma}^{\text{eq}} \) in this equation is replaced by \( f_{p\sigma}^0 \). Any two-observable equal-time correlation function is obtained from Eq. (5.2) by integration.

For the problem of electrons scattering off impurities in a temperature gradient, which we studied in Sec. III A, it turns out that Eq. (5.2) leads to the correct result for the equal-time density-density and current-density correlation functions. The result for the equal-time current fluctuations in the presence of an electric field, with or without phonons (Secs. III B and IV), may also be obtained correctly from Eq. (5.2). This is not true, however, for the corrections of order \( (eE l / k_B T)^2 \) discussed in the appendix.

The reason for the (fortuitous) agreement in the case of the leading corrections to equal-time current fluctuations may be seen as follows: For a uniform system, one obtains from Eq. (3.5)
\[
\int_{-\infty}^{\infty} \frac{d \omega}{2\pi} \langle \tilde{\mathcal{J}} (\vec{r}_1, \omega) \tilde{\mathcal{J}} (\vec{r}_2) \rangle
\]
\[
= \int_{-\infty}^{\infty} \frac{d \omega}{2\pi} \frac{\tau_\omega^2}{\omega^2 + (\tau_\omega)^2} \langle \tilde{\mathcal{J}} (\vec{r}_1, \omega) \tilde{\mathcal{J}} (\vec{r}_2) \rangle .
\] (5.3)

For the leading corrections to the current fluctuations, the complete formula (3.7) for the Langevin-force autocorrelation function simplifies to the form (3.13a). Substituting that equation in Eq. (5.3) one finds
\[
\int_{-\infty}^{\infty} \frac{d \omega}{2\pi} \langle \tilde{\mathcal{J}} (\vec{r}_1, \omega) \tilde{\mathcal{J}} (\vec{r}_2) \rangle
\]
\[
= \delta^2 (\vec{r}_1 - \vec{r}_2) \sum_p \langle \tilde{\mathcal{J}} \tilde{\mathcal{J}} \rangle_p (1 - f_{p}^0) ,
\] (5.4)

which is identical to the result which may be obtained to leading order from Eq. (5.2). For higher-order corrections in powers of \( q / l \), the simplification (3.13a) cannot in general be done; hence entropy considerations give the wrong result.

As Lax has demonstrated,\(^{30}\) in the case of one-body collision processes, Eq. (5.2) will, in general, give the wrong result for equal-time fluctuations unless the stationary distribution function satisfies detailed balance \( (J_{p\sigma} = J_{p\sigma}^0) \), or unless Boltzmann statistics may be used. The latter special case has also been noticed by Gantsevitch et al.,\(^{11}\) who also pointed out that for a nonequilibrium classical gas with two-body collisions, entropy considerations are in general useless.

In nonequilibrium steady states both the stationary distribution function and the fluctuations, in general, depend on the details of the random collisions. In equilibrium, by contrast, collisions do not appear explicitly in neither the distribution function (even though one uses them to justify the existence of equilibrium in the first place), nor in the calculation of equal-time fluctuations (entropy considerations are sufficient) because of detailed balance.

VI. CONCLUSION

We have elucidated the physical meaning of previous sophisticated calculations of fluctuations in certain dissipative steady states by reproducing these results in detail with simpler methods. For the problems we have studied, a local equilibrium extension of the Langevin formalism at the hydrodynamic level works to only about 10\% accuracy when the isotropic part of the stationary distribution function differs from a local equilibrium form. A Langevin-type formalism, however, remains valid and there exist formulas for the Langevin-force correlation which depend on the detailed shape of the stationary distribution function.

Our main result is a realistic calculation of the high-frequency current fluctuations induced by a steady electric field applied to a thin metallic film. At low temperature and under appropriate conditions, these additional fluctuations give a simple way to measure an inelastic relaxation time in the film, in a regime where it cannot easily be measured otherwise because the usual transport coefficients and equilibrium fluctuations are mainly determined by elastic scattering. Physically, the additional fluctuations can easily be understood as arising from an effective electronic temperature rise determined by the balance between the Joule heating rate and the energy relaxation rate, the phonons remaining at the bath temperature. The detailed shape of the isotropic part of the electronic distribution function is, however, not well described by a local temperature. Previous calculations\(^{31}\) of the phonon spectrum generated by Joule heating in thin metallic films have assumed that the electronic distribution function can be described by a local temperature. For the specific case we have considered this may be a questionable hypothesis. Our calculation is, however, not appli-
cable to thicker films or higher levels of excitations where electron-electron collisions may become important.

A quantitative experimental verification of our result, Eq. (4.20), would be most important from the point of view of nonequilibrium statistical mechanics. And if our prediction is valid, it could be generalized and then profitably used in the following fields of condensed-matter physics:

(a) In nonequilibrium superconductivity, the normal-state inelastic scattering time $\tau_0$ [see Eqs. (3.24) and (4.24)] appears as a parameter in many of the results, and hence can be indirectly measured in the superconducting state. Our calculation provides a way of obtaining this inelastic scattering time directly in the normal state.

(b) In the field of low-temperature refrigeration, it has been known for some time that in sufficiently small metallic systems, the thermal impedance between the electrons and phonons becomes an important mechanism limiting the efficiency of heat transfer. If we define the thermal resistance as $R_T = \delta T/Q$, where $Q = \sigma E^2 V$, one finds that nonequilibrium current fluctuations [Eq. (4.21)] provide a way of measuring $\delta T$, and hence the thermal resistance between electrons and phonons. The theoretical result that one obtains for $R_T$, using Eq. (4.17) for $\delta T$ and $\sigma E^2 V$ for $Q$, differs only by a numerical factor from that obtained in earlier calculations. That numerical factor is not equal to unity because the nonequilibrium situation considered here is different from the usual one and because there is some ambiguity in the definition of $\delta T$. Given the widespread use of metal sintering in the refrigeration process, this electron-phonon thermal resistance is very relevant for low-temperature experimentation. Anderson and Peterson have measured this thermal resistance with a different technique.

(c) Finally, it is interesting to speculate that if results analogous to those we have derived hold in the localized regime of two-dimensional metallic films, high-frequency nonequilibrium current fluctuations may provide a way to independently determine the inelastic scattering rate which arises as a parameter determining the logarithmic temperature dependence of the resistance in the localization theory of Abrahams et al. In view of the existence of a competing theory for this logarithmic temperature dependence of resistance, it may be important to have this additional way to estimate the inelastic relaxation rate to remove one adjustable parameter in the theory.

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APPENDIX

In this appendix, we compute corrections of order $(eE/k_B T)^2$ to the current fluctuations in the presence of an electric field and a self-consistent temperature gradient (see Sec. III B). It is clear that these corrections are smaller than those already computed by a factor of order $(q l)^2$, where $q \sim d^{-1}$, hence they are not experimentally relevant. We are rather interested in showing that the corrections computed in Ref. 17 using Green's functions can also be obtained by the simpler techniques of this paper.

Figure 2(a) is representative of four diagrams of Fig. 4 in Ref. 17 which have the property of vanishing when the applied electric field and the measured current are perpendicular. The terms which have a similar property in the present calculation come from the following second-order contribution to Eq. (3.7):

$$\langle \xi(t_1, t_2) \xi(t_{1'}, t_{2'}) \rangle_{ne} = V^{-1} \delta^3(\vec{r}_1 - \vec{r}_2) \delta(t_1 - t_2) 2 \tau_e^2 \times \sum_{\rho_1 \rho_2} \left( -\nu_{\rho_1} \nu_{\rho_2} \right) W_{\rho_1 \rho_2} \left( -2 J_{14}^{(1)}, J_{14}^{(1)} \right).$$

Using Eq. (3.9), one finds from this equation

FIG. 2. Diagrams representative of certain classes of diagrams of Ref. 17. Current vertices are marked by $\pm \nu_0$ and electric field vertices by $\pm \nu$. We have not indicated the vertex corrections which vanish for isotropic scattering. See Ref. 17 for more details.
\[ S_I(\omega) = \frac{2k_BT}{R} \left[ \frac{1}{18} \frac{eE}{k_BT} \right]^2, \quad (A2) \]

which should be compared with the first term of Eq. (3.53) in Ref. 17.

Since to leading order \( f^{(2)}_p \) is proportional to \( (eE/qk_BT)^2 \) [see Eq. (3.11)], terms of order \( (eE/k_BT)^2 \) can also be found by including higher powers of \( qI \) in the stationary solution of the Boltzmann equation, as well as in the solution to the linearized equation (3.2) for the fluctuations.

From the latter equation, it is easy to find, in the limit \( \omega \tau_e << 1 \),

\[ \delta f_p(q, \omega) = \frac{\delta f_p(q, \omega) + \delta f_p(q, \omega)}{iqv^+_\tau_e + 1} + \frac{\delta f_p(q, \omega)}{iqv^-_\tau_e + 1}, \quad (A3) \]

where a bar denotes angular average

\[ \delta \bar{f}_p = \int \frac{dp}{4\pi} \delta f_p, \quad (A4) \]

and \( v^\pm_\ell \) is the projection of the Fermi velocity in the direction \( \ell \) of the temperature gradient. Using Eq. (3.4) for the current and Eq. (A3) for \( \delta f_p \), we can compute the correlation function which was evaluated in Ref. 17 and which contained ambiguities parametrized by the factors \( \alpha \) and \( \beta \). Assuming that the current is measured in the direction \( l \) perpendicular to \( i \), we find that \( \delta \bar{f}_p \) does not contribute to the correlation function and we are left with

\[ (j'(q(1+\alpha)/2, \omega)j'(q(1-\alpha)/2, -\omega)) = \frac{1}{V^2} \sum_{p_1 \sigma, p_2 \sigma_2} v^\ell_\sigma v^{i\prime}_\sigma (\delta J_{p_1}(q(1+\alpha)/2, \omega)\delta J_{p_2}(q(1-\alpha)/2, -\omega)) \]

\[ \times \frac{[iqv^+_\tau_e(1+\alpha)/2+1/\tau_e][iqv^-_\tau_e(1-\alpha)/2+1/\tau_e]}{[iqv^+_\tau_e(1+\alpha)/2+1/\tau_e][iqv^-_\tau_e(1-\alpha)/2+1/\tau_e]} \]

\[ (A5) \]

To evaluate \( \delta J_{p_1} \delta J_{p_2} \), we use Eq. (2.8) and again the fact that the directions \( i \) and \( l \) are perpendicular:

\[ (j'(q(1+\alpha)/2, \omega)j'(q(1-\alpha)/2, -\omega)) = -2\mathcal{T} \int \frac{d^3p}{(2\pi)^3} \frac{(1-2f^0_p)[f^{(2)}_p(q) + f^{(2)}_p(q)]\Delta^\tau_{\ell} f^{(1)}_{p_2} f^{(1)}_{p_1}}{[qv^+_\tau_e(1+\alpha)/2-i/\tau_e][qv^-_\tau_e(1-\alpha)/2-i/\tau_e]} \]

\[ (A6) \]

\( \mathcal{T} \) is the time interval over which the time Fourier transforms are taken. The factor \( \alpha \) symbolically represents the fact that all correlation functions of the type \( (j'(q/2-k, \omega)j'(q/2+k, -\omega)) \) are nonvanishing for this problem. The real-space correlation function, which is measured in experiments, can be computed with the help of all these nonvanishing Fourier-space correlation functions. \( \delta \bar{f}_p \)

To evaluate \( f^{(2)}_p \) to higher order in \( qI \), we first notice that \( f^{(1)}_p \) obeys the equation,

\[ i\Delta(\bar{k}) \]

\( \Delta(\bar{k}) \)

where \( \Delta(\bar{k}) \) is the Fourier transform of a function which is unity in the region where the electric field is applied and zero outside. (This corresponds to an analytically useful idealization made in Ref. 17; there it was assumed that the system extends to infinity but that the electric field is applied to a finite region which defines the resistor of experimental interest.) The right-hand side of the second-order Boltzmann equation (3.10) is \( e\bar{E} \cdot \nabla f^{(1)}_p(\bar{r}) \); hence Fourier transforming, using the solution of Eq. (A7), and symmetrizing, one finds

\[ (iqv^+_\tau_e(1+\alpha)/2+1/\tau_e)[qv^+_\tau_e(1+\alpha)/2-i/\tau_e][qv^-_\tau_e(1-\alpha)/2-i/\tau_e] \]

\[ (A8) \]

To make contact with Ref. 17, we remove the integral over \( \bar{r} \) in Eq. (A8) and set \( \bar{k} = (1/2+\beta)\bar{q} \). Then, the right-hand side of Eq. (A8), \( \tau_e^{-1} A_p \), becomes
\[
\tau^{-1}_e A_p = \frac{e^2}{2} (e E v_p \tau) \Delta [q(1/2 + \beta)] \Delta [q(1/2 - \beta)] \\
\times \left[ \frac{[q(1/2 + \beta) + i/\tau_e] - [q(1/2 - 1/2) - i/\tau_e]}{\tau} \right].
\]

(A9)

Equation (A8) may be solved to yield

\[
J_{p^2}^{(2)} + J_{p^2}^{(2)} = (1 + h_p) \int \frac{d\phi}{4\pi} h_p A_p \\
+ \frac{1}{1 - \int \frac{d\phi}{4\pi} h_p} h_p A_p,
\]

(A10a)

where

\[
h_p = \frac{-i/\tau_e}{q v_p - i/\tau_e}.
\]

(A10b)

The first term on the right-hand side of Eq. (A10a) is, to leading order, proportional to \(q^{-2}\) as found in Eq. (3.11). The last term, by contrast, directly leads to contributions of order \((eE1/k_B T)^2\) when substituted in Eq. (A6) with \(q = 0\):

\[
-2\mathcal{F} \int \frac{d^2 p}{(2\pi)^3} (1 - 2f_p^0) (e E v_p \tau) \frac{\partial^2 f_0}{\partial \epsilon_p^2} [q v_p^l(1 + \alpha)/2 - i/\tau_e] \int \frac{d\phi}{4\pi} h_p A_p.
\]

(A11)

This gives the following contribution to the current fluctuations:

\[
S_f(\omega) = \frac{2k_B T}{R} \left[ \frac{e E I}{k_B T} \right]^2.
\]

(A12)

One can check that this result becomes a factor of \(1/3\) smaller when the measured current is perpendicular to the electric field. Thus, the contribution Eq. (A12) is identical to the second term of Eq. (3.53) in Ref. 17 which came from diagrams having the topology illustrated in Fig. 2(b), and hence, the same dependence on the relative orientation of electric field and measured current.

Substituting the rest of Eq. (A10) in Eq. (A6), one finds the following additional contribution:

\[
\langle j^l(q(1 + \alpha)/2, \omega) j^l(q(1 - \alpha)/2, -\omega) \rangle_{ne}
\]

\[
= -2\mathcal{F} \int \frac{d^2 p}{(2\pi)^3} \frac{v_p^l q - 2i/\tau_e}{v_p^l q - i/\tau_e} \frac{[q v_p^l(1 + \alpha)/2 - i/\tau_e]}{[q v_p^l(1 - \alpha)/2 - i/\tau_e]} \int \frac{d\phi}{4\pi} h_p A_p \\
+ \frac{1}{1 - \int \frac{d\phi}{4\pi} h_p} h_p A_p,
\]

(A13)

which is exactly the result which can be found for the diagrams having the topology illustrated in Fig. 2(c). The comparison is most easily made when all the integrations over \(\epsilon_p\) have been done in the diagrams but not the integrations over \(\omega\) or over the direction of momentum. Note that the factor

\[
\int \frac{d\phi}{4\pi} h_p A_p
\]

in Eq. (A13) is the inhomogeneous term of the integral equation for the vertex \(\Gamma\) in Ref. 17.


8R. Brout, Physica XXII, 509 (1956).


18Ref. 16, Appendix C.


20R. E. Prange and L. P. Kadanoff, Phys. Rev. 134, A566 (1964), have shown this to be true for the electron-phonon system. The methods of Ref. 17 show that even if $\hbar/\tau_c \gg k_B T$, ordinary transport theory is also valid for impurity scattering. We surmise that this is also the case for combined electron-phonon and electron-impurity scattering. See, for example, the normal-state limit of the kinetic equations of A. Schmid and G. Schön, J. Low Temp. Phys. 20, 207 (1975).


23Onuki, in Ref. 5, discusses when the Gaussian limit is recovered.

24The voltage fluctuations in the presence of a constant current are related to the result (3.16) by a factor $R^2$.

25This result remains valid even if the impurity scattering is not isotropic as we have assumed here.

26This is not hard to prove from the fact that the order $E^3$ corrections to the Boltzmann are odd in $\varepsilon_0$. This was also proved by diagrammatic methods: A.-M. Tremblay, Ph.D. Thesis, MIT, 1978 (unpublished), Chap. IV, Appendix F.


28Ref. 19, Sec. 7.5.1.


30See Ref. 2, Sec. 12.


32S. B. Kaplan, C. C. Chi, D. N. Langenberg, J.-J. Chang, S. Jafarey, and D. J. Scalapino, Phys. Rev. B 14, 4854 (1976); 15, 3567(E) (1977). Note that $\tau_\varepsilon$ defined in this reference is smaller than that defined in Eq. (3.24) by a factor $7\xi(3) \approx 8.4$.


35For a good review, see J. P. Harrison, J. Low Temp Phys. 27, 467 (1979).


39See the derivation of Eq. (3.16) and Ref. 16, Appendix A, for an example of how this can be done.