Negative moments of currents in percolating resistor networks

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(Received 5 August 1988; revised manuscript received 15 June 1989)

It has been shown that the positive-integer moments of the current distribution in a percolating resistor network theoretically suffice to determine that distribution and hence all of its moments. We discuss the inherent numerical and analytical difficulties involved when the negative moments are reconstructed from the positive ones.

In a recent paper[1] a new program of characterization of universal properties of multifractal moments was suggested. It focuses on universal probability distributions for positive integer moments M_q, and on the corresponding universal exponents and amplitude ratios. This description is based on analogies with critical phenomena. Its completeness hinges on a theorem by Hausdorff and Bernstein which states that an exact knowledge of the positive-integer moments of a probability distribution on a finite support, suffices to characterize the entire distribution. The usual study of multifractals,[2] on the other hand, does not address the question of amplitude ratios but includes instead a discussion of negative moments. Although these quantities generally exhibit less universal behavior than the positive moments, physical situations may require their evaluation. It was stated in Ref. 1 that, *in principle*, it is possible to construct *negative* moments from the knowledge of the positive ones, although in practice such a procedure might not be desirable. The purpose of this Brief Report is to explicitly show that indeed the evaluation of the negative moments of currents in percolating systems from the positive ones involves severe practical difficulties. That this can be the case is suggested by previous work[3] where the negative moments had been analyzed directly for percolating resistor networks and found to display a type of Lifshitz[4] phenomenon, wherein, for $q<0$, M_q is dominated by unusually small currents in extremely low-probability configurations.[5] Since Lifshitz phenomena are notoriously difficult to investigate numerically, this raises the question of whether numerical or analytic[6] calculations of the positive moments can be used in this way to obtain the negative moments. This note explicitly shows why, when negative moments are required, a direct evaluation is preferable to a calculation from the positive-integer moments.

We consider a percolating system in which nodes are connected by resistors, each of which randomly assumes the values 1 and 0 with respective probabilities p and $1-p$. If a unit current is inserted into the network of resistors at node $\mathbf{x}$ and removed at node $\mathbf{x}'$, we define

$$M_q(\mathbf{x},\mathbf{x}') = \left[\sum_b i_b^{2q} \Big/ \sum_b 1 \right], \tag{1}$$

where the sums run over all bonds with $i_b \neq 0$ and $[\cdots]$ indicates an average over all configurations. Series techniques involve calculations of a quantity essentially equivalent to $M_q(p) \equiv \sum_{\mathbf{x}'} M_q(\mathbf{x},\mathbf{x}')$. Here we also consider $M_q(\mathbf{x},\mathbf{x}')$ for a system of size L when $|\mathbf{x}-\mathbf{x}'|$ is of order L and we denote this quantity $M_q(L)$. For $q>0$, one has, at the percolation threshold, in the asymptotic limit of large L,

$$M_q(L) = A_q L^{x_0 - x_q}, \tag{2}$$

where A_q are nonuniversal amplitudes and $-x_q$ are the multifractal exponents, with $-x_0$ the fractal dimension of the backbone. For negative q, it was found[3] that the threshold $p_c(q)$, at which M_q diverges, decreases as q becomes more negative. This was attributed to the fact that the small currents which dominate the negative moments, depend exponentially on L. In fact, the small currents arise, for example, from long "ladders" (see Fig. 1 of Ref.

3). The current in the last rung of a ladder of l rungs is of order $\exp(-\alpha l/2)$, contributing an amount of order[3]

$$\delta M_q \sim \exp(-\alpha q l)\exp[-\beta(p)l] , \quad (3)$$

where $\exp[-\beta(p)l]$ is the probability that such a configuration occurs. Since $l \sim L^y$ with $y \geq 1$,[3] δM_q increases with L exponentially (or even faster) for $q<0$ and $\alpha|q|>\beta(p)$.

Here we wish to point out that exponential contributions as in (3) prevent, in practice, the determination of the negative moments from the knowledge of only the positive ones due to intrinsic difficulties in obtaining these moments to a sufficiently high accuracy. For example, let us consider the possibility mentioned in Ref. 1 that the ϵ expansion results might be used to evaluate the negative moments. This expansion is basically a perturbative technique (albeit a fancy one) in which the iterative length rescaling is designed to identify a power-law behavior as in (1). Weak essential singularities due to exponential contributions as in (3) are not accessible to these methods. Such a behavior is analogous to the Griffiths singularities[7] in dilute ferromagnets or the Lifshitz phenomenon[4] at the band tails in the density of states of an electron in a bounded random potential. No one has yet devised a scheme to observe singularities in the dilute system which occur at the transition temperature of the pure system due to rare undiluted regions. These effects, and the ones that are indicated by (3) for the positive moments, correspond to *unobservable* exponential singularities.[8]

The accuracy required to obtain the negative moments from the positive ones may be estimated from the Hausdorff-Bernstein reconstruction formula.[1,9] We first state the results, and then proceed to prove them. Suppose that the smallest current in the network is a fraction $\exp(-\alpha L^y)$ of the input current. To recover, with N positive-integer moments, the leading contribution to the negative moment of order $q<0$, one needs $N \gg q$, $N \gg \exp(\alpha L^y)$ and, assuming $N \simeq \exp(\alpha L^y)$, a relative error in the positive moments which is less than $\exp \times[-\gamma \exp(\alpha L^y)]$ where $\gamma \simeq \ln 3$. Since, at best, one may obtain power-law corrections to scaling, it is clear that such an accuracy is unattainable in analytic calculations. Likewise, in numerical calculations, the required precision, even for $\alpha L^y = 5$, exceeds the 16 bytes accuracy of commercial computers.

To obtain the above estimates, we first employ the reconstruction formula,[1,9] which states that given the first N positive moments, one has

$$\lim_{N\to\infty} \int_0^1 di^2 P_N(i^2,L) f(i^2) = \int_0^1 di^2 P(i^2,L) f(i^2) , \quad (4)$$

where the right-hand side exists. Here $P_N(i^2,L)$ is defined via the first N moments of $P(i^2,L)$,

$$P_N(i^2,L) \equiv \sum_{k=1}^{N} C_k^N \langle i^{2k}(1-i^2)^{N-k}\rangle \delta(i^2 - k/N) , \quad (5)$$

where $C_k^N \equiv N!/k!(N-k)!$, the angular brackets denote an average over the true distribution of currents, and $f(i^2)$ is continuous over (0,1]. Now we single out the smallest current, which we assume to dominate the negative moments in the manner indicated by (3),

$$P_N(i^2,L) = g_N(i^2,L) + a e^{-\beta l}\delta(i^2 - e^{-\alpha l}) . \quad (6)$$

Here g_N leads to power-law scaling for M_q as in (2) and a is of order unity and will be dropped subsequently.

The first question we consider is how large N needs to be so that (4) can be used for an accurate estimation of the negative moments. Using (4)–(6), we have for the $-q$th moment, M_{-q} $(q>0)$,

$$\begin{aligned} M_{-q} &= \int_0^l di^2 P_N(i^2,L) i^{-2q} \\ &= R + e^{-\beta l} \sum_{k=1}^{N} C_k^N (k/N)^{-q} e^{-k\alpha l}(1-e^{-\alpha l})^{N-k} , \end{aligned} \quad (7)$$

where R stands for the "regular" contribution from g_N. For large values of N we can estimate the sum by its maximal term. Assuming $N \gg q$ and using Stirling's formula, we find a maximum at $k = k_{\max} = N\exp(-\alpha l)$. Substituting this result into (7), we obtain

$$M_{-q} = R + A e^{(q\alpha - \beta)l} , \quad (8)$$

which is the expected behavior. The conditions that must be fulfilled to obtain (8) are met when $N = k_{\max}\exp(\alpha l) \gg \exp(\alpha l)$. It follows that an exponentially large number of positive moments are needed for retrieving the contribution of exponentially small currents to negative moments.

The second question we address is the following: Assuming N is large enough to meet the above condition, to what precision does one need the positive moments in order to retain the information about the negative ones. Using (4) and expanding the averaged product in (5), we have

$$M_{-q} = \sum_{k=1}^{N} C_k^N (k/N)^{-q} \sum_{r=1}^{N-k} C_r^{N-k}(-1)^r M_{k+r} . \quad (9)$$

Assuming every moment has an error

$$M_{k+r} = M_{k+r}^0 + \delta M_{k+r} , \quad (10)$$

and also assuming that most of the error in M_{-q} comes from the neighborhood of r_0, where C_r^{N-k} is sharply peaked, we use Stirling's formula to find

$$\sum_{r=1}^{N-k} C_r^{N-k}(-1)^r \delta M_{k+r} \simeq 2^{N-k}\delta M_{k+r_0} . \quad (11)$$

Substituting (11) into (9) and approximating the sum by its maximal term (for which $k \simeq N/3$) we have

$$\delta M_{-q} \simeq 3^{N+q}\delta M_{k_0+r_0} .$$

To obtain $\delta M_{-q} \ll M_{-q}$ we thus require

$$(\delta M_{k_0+r_0})/(M_{k_0+r_0}) \ll 3^{-(N+q)}(M_{-q})/(M_{k_0+r_0}) . \quad (12)$$

Choosing $N \simeq \exp(\alpha l)$ (to set the most relaxed condition) and inserting (8) and $M_{k_0+r_0} \sim 1$ on the right-hand side of (12) we have

$$\begin{aligned} \delta M_{k_0+r_0}/M_{k_0+r_0} &\ll e^{-(q+e^{\alpha l})\ln 3}(R + A e^{(q\alpha-\beta)l}) \\ &\simeq A e^{-e^{\alpha l}\ln 3} , \end{aligned} \quad (13)$$

where we assumed $N > \exp(\alpha l) \gg (q\alpha - \beta)l + q\ln 3$. Relation (13) is the result quoted above. Since in real (or simulation) measurements the relative error increases with the order of the (positive) moment, then (13) sets the most relaxed condition for all $M_{k<k_0+r_0}$.

As a final remark, we note that the q dependence of $p_c(q)$ found in Ref. 3 is a novel direct measurement of exponentially rare terms like those of (3). Indeed, we expect similar q-dependent thresholds for negative moments of appropriately defined densities of states of localized wave functions.

We acknowledge the support of the U.S.-Israel Binational Science Foundation (BSF) and the Israel Academy of Sciences and Humanities (at Tel Aviv), the Natural Sciences and Engineering Research Council of Canada and the Steacie Foundation (at Sherbrooke), the National Science Foundation (Grant No. 85-20272) (at Pennsylvania), and the Canadian Summer Institute in Theoretical Physics, where this work was done.

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