Amplitudes of Multifractal Moments at the Onset of Chaos: Universal Ratios and Crossover Functions

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A new program of characterization of multifractal moments is proposed. For the circle map, the usual multifractal moments are described by both scaling exponents and amplitudes. These amplitudes depend sensitively on the starting point of the time series used to define these moments. This leads naturally to a statistical description which is *universal* and analogous to that used in critical phenomena and in other fields where multifractals occur. This description can be used not only at the critical point, but also in the crossover region where the analog of the correlation length is large but not infinite.

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Present understanding of the transition from regular to chaotic behavior is based on the application of renormalization-group and scaling concepts to simple dynamical systems or maps. In the approach pioneered by Feigenbaum, the trajectory in phase space is characterized by universal but very irregular scaling functions, or by universal spectra which are discontinuous everywhere. The theoretical and experimental inconvenience of discontinuous functions motivated the development of the multifractal approach,[1] which concentrates on a smooth function, the so-called $f(\alpha)$ spectrum. However, this approach appears to depart from the usual critical phenomenology since it emphasizes a *continuous* set of exponents. After the anticipatory work of Mandelbrot, the multifractal formalism appeared simultaneously in many fields concerned with fractals.[2]

In this Letter, we propose a new program of multifractal characterization based on a countable infinite set of multifractal moments. Our approach gives a smooth description of the *universal* information contained in not only exponents, but also amplitudes of the multifractal moments. These amplitudes depend sensitively on the starting point of the time series which defines the multifractal moments. A possible description of this dependence would be through Fourier transforms. (The Fourier transform of the amplitude of the first moment is, in fact, proportional to the usual spectrum.) We adopt instead a smooth statistical description of the complete set of multifractal moments which should be more convenient for experiment than descriptions based on highly irregular functions. Moreover, our statistical approach is very close in spirit to critical phenomena. This has two advantages. First, it provides a unified theoretical perspective on critical phenomena and multifractals in various fields.[3] (We have already presented the analog of the present results in the context of percolation,[4] and it is clear that it generalizes to most cases where multifractal behavior has been observed.) Second, the similarity with critical phenomena leads to practical suggestions for analysis of experimental data,[5] namely, in addition to exponents, there are other measurable universal quantities, analogous to universal amplitude ratios,[6] at the critical point. There exists also a natural definition of the analog of the correlation length. And, finally, it is possible to study the crossover region near criticality within the same approach.

To illustrate our point of view, let us concentrate on the circle map

$$\theta_{i+1}=f(\theta_i)\equiv\theta_i+\Omega-(K/2\pi)\sin(2\pi\theta_i)\,, \tag{1}$$

where K and the bare winding number Ω, or K and the period P can be chosen as the control parameters. We are interested in properties at and near the critical point corresponding to a mean winding number ρ equal to the golden mean σ $[\rho=\sigma=(\sqrt{5}-1)/2]$. The functional renormalization group of Feigenbaum, Kadanoff, and Shenker[7] and of Rand and co-workers[8] demonstrates that there exists universal properties valid for a wide class of maps with a cubic inflection point at criticality. Although the first studies concentrated on the successive closest returns near the origin, the multifractal approach of Ref. 1 characterizes the successive closest returns of *any point* of the trajectory. Closest returns occur at Fibonacci time scales F_n whose ratios approximate the golden mean ($F_n=F_{n-1}+F_{n-2}$, $F_1=F_2=1$, $F_{n+1}\sim\sigma^{-n}$). Given a starting point x_1, Ref. 1 shows that the following multifractal moments

$$M_q(F_n,x_1)=\frac{1}{F_{n+1}}\sum_{1\le i\le F_{n+1}}|\hat{f}^{(F_n)}(x_i)-x_i|^q \approx C_q(F_n,x_1)F_{n+1}^{-[\tau(q)+1]} \tag{2}$$

scale with F_n, the analog of the finite time at which the system is probed,

$$\hat{f}^{(F_n)}(x)\equiv f^{(F_n)}(x)-F_{n-1},\quad x_i\equiv f^{(i-1)}(x_1)\,,$$

$$f^{(i)}\equiv f\circ f\circ f\cdots\circ f,\ i\text{ times}\,.$$

In (2), the amplitudes $C_q(F_n,x_1)$ are weakly dependent on the order of the moment q and their variation with F_n is bounded. However, these amplitudes depend sensitively on the starting point x_1, which in Ref. 1 is always chosen as the cubic inflection point. This choice is *a priori* arbitrary since, at criticality, ρ is irrational and that implies that the orbit is dense. We demonstrate below that there is no self-averaging and that the amplitudes of the multifractal moments exhibit large variations as a function of the starting point x_1 of the time series used to define them. The relative fluctuations are *critical*, i.e., independent of the time scale F_n at which the system is probed. The easiest way to experimentally access such ill-behaved functions is through their statistical properties.

Consider then the multifractal moments (2) as a set of random variables. The *a priori* probabilities determining the statistical ensemble are those chosen for the starting point x_1 of the iterations. The choice of this probability distribution is irrelevant. Indeed, the multifractal moments are invariant under the dynamics of the map [i.e., $M_q(F_n,x_1)=M_q(F_n,f(x_1))$ for $n\gg 1$] and, as a consequence, all averages of the multifractal moments are reduced to the average with respect to the invariant measure associated with the map. We can thus define unambiguously the joint probability distribution $\mathcal{P}$ for positive integer moments. This $\mathcal{P}$ characterizes completely the multifractal properties since the countable infinite set of *positive integer* moments suffices to reconstruct distributions [such as that for the closest-return distances[3] $|\hat{f}^{(F_n)}(x_i)-x_i|$], whose support is bounded.[4] Near criticality, $\mathcal{P}$ is a generalized homogeneous function

TABLE I. Universal ratios defined by Eq. (4). The first column (I) is for the map defined by (1) and the second column (II) for the map $\theta_{t+1}=\theta_t+\Omega-K[\sin(2\pi\theta_t)+0.2\times\sin(6\pi\theta_t)]/2\pi$. The results are for F_{10} and 2×10^4 samples. The ratios computed for larger F_n (we have checked $n=10$ to 12) or for the above two maps are within the statistical uncertainty on the last digit (written in parentheses).

$q,r;k,l$	$A(q,r;k,l)$ I	II
1 0 2 0	0.0100(2)	0.0101(2)
1 2 1 1	0.0187(2)	0.0192(3)
1 2 2 1	−0.001 17(5)	−0.001 19(4)
1 2 1 2	−0.002 27(5)	−0.002 35(5)
2 0 2 0	0.0521(4)	0.0525(9)
0 3 0 2	0.172(1)	0.166(4)

$$\mathcal{P}(a_0M_0,a_1M_1,\ldots,a_qM_q,\ldots;a_K(K_c-K);a_PP;F_n)$$
$$=\lambda^{-\tau(0)-1}\lambda^{-\tau(1)-1}\cdots\lambda^{-\tau(q)-1}\mathcal{P}\left(\frac{a_0M_0}{\lambda^{-\tau(0)-1}},\frac{a_1M_1}{\lambda^{-\tau(1)-1}},\ldots,\frac{a_qM_q}{\lambda^{-\tau(q)-1}},\ldots;\frac{a_K(K_c-K)}{\lambda^{-1/\nu}};\frac{a_PP}{\lambda};\frac{F_n}{\lambda}\right), \quad (3)$$

which is universal apart from the metric factors $a_q,\ldots,a_K,a_P$ which are map dependent. Equation (3) is the cornerstone of our approach: It exhibits the infinite (but here discrete) set of exponents, as well as gap scaling[6] for cumulants of the variables M_q, i.e., $\langle\langle(M_q)^k\rangle\rangle\approx F_{n+1}^{-k[\tau(q)+1]}$, where double angular brackets stand for cumulant averages. Also, (3) describes not only the critical point, but also its approach through changes of the control parameters K, P, and the "finite size" F_n. Finally, note that the role as well as the scaling properties of $\mathcal{P}$ are similar to those of the free energy in critical phenomena, hence the similarity in point of view.

At criticality, (3) implies, in particular, that ratios of cumulents, such as

$$A(q,r;k,l)\equiv\frac{\langle\langle M_q(F_n,x_1)^kM_r(F_n,x_1)^l\rangle\rangle}{\langle\langle M_q(F_n,x_1)\rangle\rangle^k\langle\langle M_r(F_n,x_1)\rangle\rangle^l}, \quad (4)$$

are independent of F_n and universal. To exhibit universality, a synopsis of the numerical results is reported in Table I for two different maps on the critical manifold.

The universal ratios are a way to obtain quantitative information which would otherwise be extracted from generalizations of the usual spectrum at the onset of chaos. The universality of ratios, in fact, follows from the universality of these spectra, which can be studied following the renormalization-group approach of Kadanoff.[9,10] Let $a_q(F_n,k)$ be the Fourier coefficients of the function $M_q(F_n,h(x_1))$, where h is the homeomorphism which conjugates the map to a pure rotation with the same rotation number [i.e., $h\circ f\circ h^{-1}(x)\equiv R(x)\equiv x+\sigma$]. An example of relation between universal ratio and Fourier coefficients is

$$\frac{\langle\langle[M_q(F_n,x_1)]^2\rangle\rangle}{\langle\langle M_q(F_n,x_1)\rangle\rangle^2}=\sum_{k\neq0}\frac{|a_q(F_n,k)|^2}{a_q(F_n,0)^2}. \quad (5)$$

We can show[10] that universal values of amplitude ratios follow from the fact that the low-frequency coefficients $a_q(\omega=k\sigma\ (\mathrm{mod}\,1)\ll1)$ obey a universal scaling relation, namely,

$$a_q(\omega=\sigma^l(r\sigma-s),F_n)=\sigma^{n[\tau(q)+1]}a_q(\omega=r\sigma-s,1),\quad n\gg1, \quad (6)$$

and give the dominant contribution to ratios such as (5) in the infinite-F_n limit. The picture is that an integer k defines a frequency scale $\omega=k\sigma$ (mod 1) and that, as usual,[8] the universal self-similar character is recovered in the limit where ω tends to zero. Figure 1 corroborates (6) by showing that the periodic components of the $q=2$ multifractal-moment amplitude are self-similar from one band $[\sigma^{j+1},\sigma^j]$ to another. They are also universal,

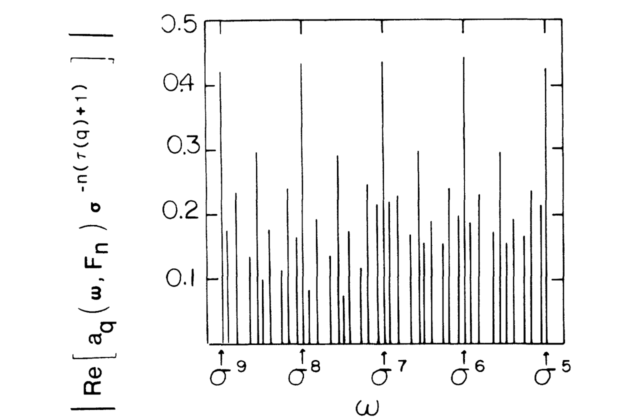

FIG. 1. Plot of $|\mathrm{Re}[a_q(\omega,F_n)\sigma^{-n[\tau(q)+1]}]|$ as a function of ω for $q=2$. The principal peaks correspond to $\omega=\sigma^m$, $m=5,6,\ldots$. ω can be written as $\sigma^l(r\sigma-s)$, where $\sigma^2<r\sigma-s<\sigma$ and $l=1,2,\ldots$. Bands are defined by the intervals $[\sigma^{l+1},\sigma^l]$, $l=1,\ldots,\infty$. As in the case of the spectrum for maps of the circle at the critical golden-mean winding number, the function depicted here is self-similar from one band to another, i.e., is independent of l for any given set of frequencies which may be written in the form $\sigma^l(r\sigma-s)$. In the limit where ω goes to zero, the function $a_q(\omega,F_n)$ is universal apart from a multiplicative factor.

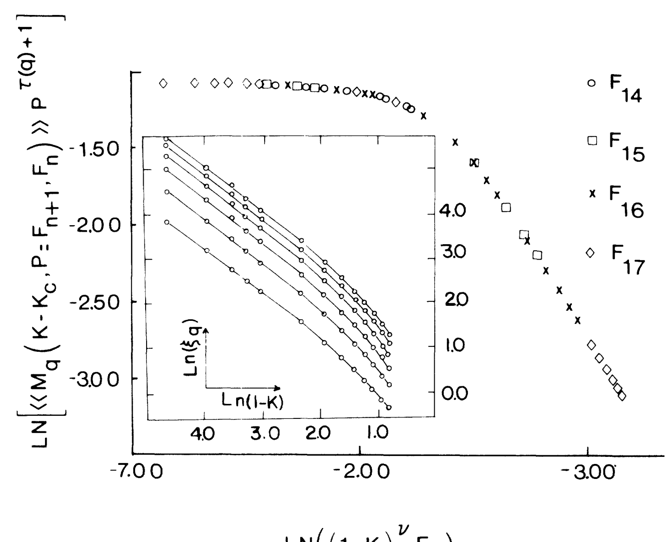

FIG. 2. Universal crossover function for the averaged multifractal moments $\langle\langle M_q(K_c-K,P=F_{n+1},F_n)\rangle\rangle$. The quantity $\ln[\langle\langle M_q(K_c-K,P=F_{n+1},F_n)\rangle\rangle P^{\tau(q)+1}/\langle\langle M_q\rangle\rangle]$ is plotted vs $\ln[(K_c-K)^\nu F_n]$ for $q=3$, $K_c=1$, values of $P=F_{n+1}$ between F_{16} and F_{18}, and $2\times10^{-7}<K_c-K<10^{-2}$. Averages are taken over 1000 starting points. All the points fall on a single curve, confirming the existence of the scaling function (8). Inset: Plot of $\ln\xi_q$ as a function of $\ln(K_c-K)$ when $\rho=\sigma$. Note that all curves are asymptotically parallel, thereby corroborating the existence of unique diverging time scale.

within an overall multiplicative constant corresponding to the projection onto the eigenvectors of the multifractal renormalization group. Taking the ratios as in (5) eliminates these nonuniversal factors. We can demonstrate,[10] in the case of $q=1$, that the Fourier coefficients are simply proportional to the usual spectrum and that nonuniversal corrections to scaling disappear when F_n goes to infinity.

In real experiments, the universal ratios may be more easily accessible than the universal spectrum of multifractal-moment amplitudes. Indeed, we find numerically that the time scale necessary to obtain reproducible peaks at σ^j in Fig. 1 is large, hence difficult to access experimentally: Peaks other than the principle ones are very sensitive to finite-size corrections even when $n=18$. By contrast, ratios of averaged low-order cumulants, such as those in Table I, vary only in the third significant digit when $n\gtrsim10$.

To discuss the crossover region in the vicinity of the critical point, the key point is that there is a *unique* characteristic time scale ξ analogous to the correlation length in critical phenomena: When the coarse-graining time F_n in (2) is much less than ξ, the system appears critical, i.e., the ratios in (4) assume their critical value, whereas, in the opposite limit, the map appears as a pure rotation ($K=0$). To define ξ, it suffices to concentrate on the two eigendirections given by the renormalization group.[7,8] In the first eigendirection, on the line $K=1$, the mean winding number is locked at the value F_n/F_{n+1} on an open interval of the bare winding number Ω. By analogy with critical phenomena, we define

$$\xi_q\equiv\sum_{k=1}^{n+1}F_k[\langle\langle M_q(F_k,x_1)^2\rangle\rangle/\langle\langle M_q(F_k,x_1)\rangle\rangle^2]\,, \qquad (7)$$

where the sum is over the different scales F_k at which one probes the system, up to the period $P=F_{n+1}$. Since the stable orbit changes continuously with the bare winding number Ω on a locking interval, the averages in (7) may be taken over a range of Ω consistent with a given period. The second eigendirection is given by a line in the Ω-K space on which the mean winding number is equal to the golden mean. Hence, there is an infinity of *a priori* starting points and the definition of ξ_q is obtained by taking P equal to infinity. Although there exists an infinite set of multifractal moments, their crossover is controlled by a *unique* characteristic time since all ξ_q diverge in the same way as one approaches the critical point, namely, as the period P on the locking intervals or as $(K_c-K)^{-\nu}$ in the other eigendirection (see inset of Fig. 2).

The crossover to the noncritical region is simply characterized by universal functions of the control parameters. These universal crossover functions are obtained as special cases from the joint probability distribution $\mathcal{P}$ in (3). They take the form

$$\langle\langle M_q(K_c-K,P=F_{n+1},F_n)\rangle\rangle \equiv P^{-\tau(q)-1}\mathcal{G}_q(D_q(K_c-K)^\nu F_n)\,. \quad (8)$$

Figure 2 verifies, for $q=3$, the scaling prediction (8) that for different F_n and K_c-K all points collapse on a single function $\mathcal{G}_q$. In the limit K goes to K_c, we recover that the multifractal moment scales as $P^{-\tau(q)-1}$. On the other hand, as P tends to infinity with $F_n/P=\sigma$ and K fixed, one goes to a pure rotation as soon as F_n is larger than ξ. In that limit, the multifractal moment scales trivially as P^{-q}, which, through $\mathcal{G}_q(y)=y^{\nu[\tau(q)+1-q]}$, implies the straight line of slope -0.5 in Fig. 2. Apart from the usual metric factors, the crossover scaling function (8) is universal, and should be accessible experimentally. The study of crossover presented here is a more general alternative to the analysis of Ref. 11 since it identifies the diverging reference time ξ, takes into account both eigendirections, and proceeds through universal crossover functions instead of parameter-dependent exponents.

In conclusion, we have shown that the study of multifractal exponents and amplitudes at the onset of chaos can be basically cast in the language of critical phenomena, namely, there is a universal fixed-point joint probability distribution for the multifractal moments and, concomitant with universality, a *unique* reference time which controls the crossover from the critical point. It has been shown that this point of view has important experimental applications since universal ratios and crossover functions are a particularly convenient way to extract new universal information on dynamical systems. A more detailed version of this work will appear elsewhere.[10]

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