Universal Properties of Multifractal Moments: Analogies with Critical Phenomena

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Continuous sets of exponents appear naturally in many fields related to fractals. Using percolation as a primary example, it is recalled that, just as in critical phenomena, the set of exponents which suffices to characterize the scaling properties is discrete instead of continuous. Analogies and differences between a specific field theory for "multicritical" behavior and one for "multifractal" behavior can then be discussed. It is shown that each exponent is associated with an operator of different symmetry. This seems to be the deeper reason behind the observability of these exponents even in situations where they would otherwise be called irrelevant. A joint probability distribution, suggested by the analogy with critical phenomena, describes the critical point as well as the crossover to the noncritical region. Measurements of this universal joint probability distribution or of the corresponding amplitude ratios should be of interest.

1. INTRODUCTION

In many fields related to fractals, it has become popular to characterize scaling properties by a continuous set of exponents [1]. In the context of percolation for example, an infinite set of exponents arises as follows [2,3]. Suppose that the conducting resistors of a percolating network are also fluctuating independently in time. The total resistance of a given network is then a random variable in time whose cumulants depend on those of each component resistor. The cumulants of a given order are assumed to be the same for all component resistors. The cumulants of the total resistance R are in principle accessible experimentally. Measurements of the second cumulant, corresponding to 1/f noise, have been performed [4]. For systems of finite size L at bulk criticality, one finds [2] for the scaling of the cumulants of order n,

$$C_R^n(L) \sim \langle M_n \rangle \sim L^{-x_n} \quad ; \qquad M_n \equiv \sum_\alpha i_\alpha^{2n} \quad , \tag{1}$$

where i is the current that flows in branch α of the time averaged network when the total input current is unity and where the brackets refer to time and disorder averages. It may seem natural also to extend the definition (1) to real values of n. In dynamical systems, the Legendre transform of the corresponding function has been interpreted geometrically as a fractal dimension [5] and has thus become widespread [6]. It has however been pointed out [7] that the positive integer values of n suffice to characterize the whole function. This follows from the Hausdorff-Bernstein reconstruction theorem [8] because (1) may be interpreted as moments of a distribution probability for the current variable i, whose range is finite: [0,1]. In other words, the system can be characterized [7] by a joint probability distribution whose scaling properties are given by

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$$\mathcal{P}(M_1, M_2, \ldots M_\infty, p-p_c, h, L) =$$
$$\lambda^{x_0+x_1+\ldots+x_\infty}\, \mathcal{P}(M_1/\lambda^{-x_1},\ M_2/\lambda^{-x_2}, \ldots M_\infty/\lambda^{-x_\infty},\ p-p_c/\lambda^{-1/\nu},\ h/\lambda^{-\phi_h/\nu},\ L/\lambda), \quad (2)$$

where p is the probability that a site or bond is occupied and h is the analog of magnetic field in percolation [10].

The formal similarity of Eq.(2) with the scaling properties of the free energy near a critical point immediately suggests [7] that $\mathcal{P}$ is a universal function of its arguments except for constant scale factors for each of these arguments. Universal amplitude ratios become of interest and should be measured. They have already been calculated in the context of percolation [7] and in the context of dynamical systems [9].

It should be clear from the preceding paragraph that analogies with critical phenomena are useful since they suggest, for multifractals, a much richer class of universal quantities to measure than had been suspected. At a more basic level however, one may wonder to what extent these analogies really hold and what the differences are. In particular one is used to the fact, in critical phenomena, that even though there is an infinite set of renormalization group eigenoperators, measurable operators in general have a non-zero projection on almost all eigenoperators, and in particular on the most relevant ones, which eventually dominate the scaling behavior. In fact, only a small number of eigenoperators are usually relevant. Most of them are irrelevant. In the case of multifractals, do we have an infinite set of relevant operators? Why don't observables couple only to the most relevant ones? It is these and related questions that we wish to answer by comparing in detail one recent study of multicritical behavior [11] and the field theory of PARK, HARRIS and LUBENSKY [12] (PHL) for multifractal moments in percolation. Note that it is the renormalization group behind this field theoretical formulation which may be seen as a justification for the universal properties of Eq.(2).

2. MULTICRITICAL BEHAVIOR IN CRITICAL PHENOMENA

We first summarize the analysis of AHARONY et al. [11] for the S_A-S_{BH} or S_C-S_I transitions. (The last two smectic phases, S_{BH} and S_I, also have hexatic symmetry). The starting Ginzburg-Landau Hamiltonian [13] is

$$H = \int d^d r \ \left\{ \ \tfrac{1}{2}|\nabla\Psi|^2 + \tfrac{1}{2}r|\Psi|^2 + u_4|\Psi|^4 + u_6|\Psi|^6 + h\mathrm{Re}(\Psi) \ \right\}, \quad (3)$$

where h vanishes in the S_A and is small in the S_C phases, while

$$\Psi = e^{6i\theta(r)} \quad (4)$$

is the local order parameter describing the six-fold symmetric orientational alignment of the lines connecting neighboring molecules in the smectic planes. u_4 is supposed larger than the value where the model is tricritical. The Hamiltonian (3) then describes the XY model. Through measurements of higher order Fourier components of the X-Ray spectrum [14] in the S_C phase, where the small h introduces long-range hexatic order, the following quantities become experimentally accessible,

$$C_{6n} \equiv \mathrm{Re} < \int d^d r \ \Psi^n(r) >. \quad (5)$$

To determine the scaling behavior of these quantities, a symmetry breaking term [11]

$$H_n \equiv g_n \int d^d r \ \mathrm{Re}(\Psi^n) \tag{6}$$

is added to the Hamiltonian (3). Asymptotically close to the XY fixed point, the free energy should scale as $F(t,g_n) = |t|^{2-\alpha} f(g_n/|t|^{\phi_n})$, where $t=(T-T_c)/T_c$, α is the XY specific heat exponent and ϕ_n the appropriate crossover exponent. Thus,

$$C_{6n} = \partial F/\partial g_n \Big|_{g_n=0} \sim |t|^{2-\alpha-\phi_n} \quad . \tag{7}$$

Using the hyperscaling relation $(2-\alpha) = d\nu$ and $|t| \sim \xi^{-1/\nu}$, where ξ is the correlation length and ν the corresponding exponent, we can rewrite (7) as,

$$C_{6n} \sim \xi^{-d} \ \xi^{\phi_n/\nu} \quad . \tag{8}$$

When hyperscaling is valid, one can interpret $(L/\xi)^d$ as the number of coherence volumes within the system of linear size L in d dimensions. Hence, for $L << \xi$, (compare (1)),

$$C_{6n} \sim L^{\phi_n/\nu} \quad . \tag{9}$$

The crossover exponents ϕ_n are not [11] a linear function of n. Several general remarks can now be made.

i) Infinite Set of Exponents (multicritical behavior): The ϕ_n form an infinite set of crossover exponents. (See also (15]). They are not [11] a linear function of n.(See following paragraph). Furthermore, they are only a subset of all possible crossover exponents, namely the subset associated with symmetry breaking perturbations. These two points are amplified below.

ii) Gap Scaling: The scaling of the usual thermodynamic observables is normally trivially obtained from a few exponents only. This is usually referred to as "gap scaling". Gap scaling also occurs in the present case. For example,

$$\langle \delta C_{6n}^{\ell} \rangle_c \equiv \partial^{\ell} F/\partial g_n^{\ell} \Big|_{g_n=0} \sim \langle \left[\int d^d r \ \mathrm{Re}(\Psi^n) \right]^{\ell} \rangle_c \sim \xi^{-d} \ \xi^{\ell\phi_n/\nu} \ , \tag{10}$$

where the subscript c indicates cumulant average. In other words, the exponents describing the scaling of the susceptibilities associated with the basic variables C_{6n} are all linearly related to the basic exponent ϕ_n. The same is true of cross-correlations. For example, for systems of size $L << \xi$, we have $\langle \delta C_6^{\ell} \delta C_{12}^{m} \rangle_c \sim L^{\ell\phi_1/\nu + m\phi_2/\nu}$. (Compare Ref.[7])

iii) Crossover Exponents Associated with Symmetry-Breaking Perturbations: The original XY model is invariant under *global* gauge transformations (rotations in the two-dimensional order parameter space $\Psi = \Psi_1 + i\Psi_2$). The crossover exponents are associated with the "symmetry-breaking" perturbations in (6). Indeed, successive terms $\mathrm{Re}(\Psi^n)$ only have an n-fold symmetry axis. With larger values of n, $\mathrm{Re}(\Psi^n)$ transforms with successively higher-dimensional representations of the rotation group O(2). In the usual language, n=2,3,4 correspond respectively to uniaxial, Potts and cubic anisotropies [11].

iv) Relevance and Irrelevance: The perturbations $|g_n| > 0$ are relevant if $\phi_n > 0$ and irrelevant if $\phi_n < 0$. In other words, if $|g_n| > 0$ for a value of n for which $\phi_n > 0$ ($n \le 3$ in the present case), the system crosses over from the vicinity of the XY fixed point to another lower-symmetry fixed point.

v) Observability of the Crossover Exponents: Here the ϕ_n are not all relevant exponents. In fact, for $n \ge 4$, they correspond to irrelevant operators.[11,15] While they are only a subset of all possible irrelevant exponents, they are, however, special because, for increasing values of n, they represent the leading scaling behavior of operators with lower and lower symmetry. In other words, while the $C_{\theta n}$ are not eigenoperators in general, they do not couple back to more relevant eigenoperators because of their symmetry properties.

In closing this section, note that even if h is strictly zero, as in the Smectic A phase, the crossover exponents would in principle be accessible through measurements of correlations functions such as

$$< \left[Re(\Psi^n(x))\right] \left[Re(\Psi^n(0))\right] > \sim x^{2\phi_n/\nu - 2d} \quad . \qquad (x << \xi) \tag{11}$$

It is the analogous type of correlation function which occurs most naturally in the field theory for the noisy percolating network [11].

3. MULTIFRACTAL BEHAVIOR ON PERCOLATING NETWORKS

The problem has been described in the introduction. In the field theory [12] of PHL for this problem, an n dimensional replica space, with components labeled α, is introduced to perform the equivalent of the cumulant averages of the noise for a given percolating network. Then, the average over different realizations of the percolating network is performed through an m dimensional replication of this n dimensional replica space. Each replication is labeled by an additional index β. The m and $n \to 0$ limits are to be taken as usual at the end of the calculation.

The derivation of the PHL field theory is lengthy, but all details may be found in Refs.[16] and [12]. Equations (3.4b),(3.19),(3.20) of Ref.[12] and (3.23 to 3.28b) of Ref.[16] show that, within unimportant constants, the generating function is

$$G_k(x,x') = <\Phi_k(x)\Phi_{-k}(x')> , \tag{12}$$

where the brackets refer to traces over the $\Phi_k(x)$ fields with the following action L:

$$L = \int d^dx \left[\frac{1}{2} \sum_{k\neq 0} r_k\Phi_k(x)\Phi_{-k}(x) + \nabla\Phi_k(x)\nabla\Phi_{-k}(x) \right] \Delta k$$
$$- \frac{1}{3} u_3 \int d^dx \left[\sum_{k_1,k_2,k_1+k_2\neq 0} \Phi_{k_1}(x)\Phi_{k_2}(x)\Phi_{-k_1-k_2}(x)\right]\Delta k_1 \Delta k_2 + \dots \; . \tag{13}$$

The k are Fourier variables whose components are labeled $k_{\alpha\beta}$ in the mn dimensional replica space. Note that despite the notation, the k are tensors of rank one and not two, as far as rotations in replica space are concerned. For each replica, the components of k can take s discrete values. There are then $(s^{nm}-1)$ fields $\Phi_k(x)$ at each spatial point x [the $k = 0$ case is omitted). Note that at the end of the calculation, the limits $n \to 0$, $m \to 0$ are taken first, then $s \to \infty$. In (13),

$$r_k = r_0 + \sum_{m=1}^{\infty} w_m (k^2)^m + \sum_{\ell=2}^{\infty} v_\ell \, K_\ell(k) \; , \tag{14}$$

where $r_0 = (p - p_c)$ with p_c the critical value of the dilution probability p, while w_m and v_ℓ are constants which depend on details of the model, and

$$k^2 \equiv \sum_{\alpha\beta} k_{\alpha\beta}^2 \qquad ; \qquad K_\ell(k) \equiv \frac{(-1)^\ell}{2^\ell (\ell!)} \sum_\beta \left[\sum_\alpha k_{\alpha\beta}^2 \right]^\ell . \tag{15}$$

The cumulants $C_R^\ell(x,x')$ of the resistance noise measured between two points x and x' may be obtained from Eq.(12) through derivatives of the form

$$C_R^\ell(x,x')\chi_p(x,x') = \lim_{n\to 0} \left. \frac{\partial G_k(x,x')}{\partial K_\ell(k)} \right|_{k=0} \tag{16}$$

where χ_p is the susceptibility function for percolation. In other words, since the K_ℓ are independent homogeneous polynomials of the components of k, they are used as a basis for the k dependence of the G_k in (12) and derivatives are taken with respect to this new basis. The following remarks now parallel those of the previous section.

i) Infinite Set of Exponents (multifractal behavior): From the recursion relations for the parameters entering (13) and (14) and standard scaling arguments, PHL find that (for $|x - x'| << \xi$)

$$C_R^\ell(x,x') \sim |x - x'|^{\psi_\ell/\nu} . \qquad (\psi_\ell/\nu = -x_\ell) \tag{17}$$

This infinite set of exponents is not a linear function of ℓ. Also, it is only a subset of all possible crossover exponents. HARRIS et al. [17] had first studied the infinite set of crossover exponents associated with the w_m in (14). However, since the corresponding operators all have the same symmetry as the one for $m = 1$, they all lead to simple corrections to scaling for the resistance, which is the observable associated with the case $m = 1$. Similarly, each ψ_ℓ appears as the leading exponent corresponding to a given symmetry (See iii below).

ii) Gap Scaling: Exactly the same remarks as for the XY model apply.

iii) Crossover Exponents Associated with Symmetry-Breaking Perturbations: Assuming s large enough that k is a continuous variable, the action (13) with

$v_\ell = 0$ is invariant under the *global* transformation of the fields,

$$\Phi'_k = \Phi_{Rk} , \tag{18}$$

where R is a rotation of the vector k in the replica space of dimension mn. In other words, the action transforms according to the unit representation of the group O(mn). When $v_\ell \neq 0$, only rotations within any of the n dimensional subspaces leave the action invariant. The derivative in (16) may equivalently be expressed in terms of the v_ℓ which then play the role of the g_n in the XY model. Clearly, the v_ℓ break the symmetry in replica space, and the same remark applies to them as applies to the g_n above.

iv) <u>Relevance and Irrelevance:</u> At first sight, the perturbations associated with the v_ℓ are all relevant since all the exponents ψ_ℓ are found to be larger than zero.[12] There are two important differences however with the XY case. a) There is no physical realization that we know of for the lower symmetry fixed point towards which the system rescales when one of the symmetry-breaking perturbations is different from zero. (All physical observables are derivatives evaluated at $k = 0$, i.e. they are properties of the symmetric fixed point.) b) There is an additional freedom to rescale k at each iteration which allows one to formulate the renormalization group in such a way that only a finite number of operators are relevant! Indeed, for the usual percolation fixed point, the rescaling of the fields Φ is found by choosing that the coefficient of the spatial gradient term in (13) is a constant. Since the recursion relations for u_3 and r_0 are completely independent of k, the percolation fixed point is the usual one. The scale factor for k may be chosen at will. This influences the recursion relations for the v_ℓ and hence the corresponding ψ_ℓ exponents. Since the ψ_ℓ are a decreasing function of ℓ, we may always choose the scaling dimension of k such that only a few of the ψ exponents are positive, without influencing the physics. The field theory of PHL corresponds to computing the power dissipated between points x and x' when a unit current is injected between these points, whatever the distance between x and x'. One could just as well decide to rescale at unit voltage instead of unit current, and this would correspond to multiplying k by a scale factor at each iteration.

v) <u>Observability of the Crossover Exponents:</u> See remarks for the XY case.

4. CONCLUSION

The existence of an infinite set of symmetry-breaking relevant operators near a fixed point with a continuous symmetry was probably first discovered in 1976 by BREZIN et al. [18]. Wegner in 1980 [19] applied similar ideas to localization. As discussed by CASTELLANI and PELITI [20], the "multifractal" description of wavefunctions near a localization threshold is not really different from the infinite set of crossover exponents of Wegner. Here we have contrasted one case in critical phenomena where an infinite set of exponents is observable, with one case in percolation where a multifractal description has been proposed, to again illustrate that multifractals simply arise from the existence of symmetry-breaking operators near a fixed point with a continuous symmetry. It is the symmetry which is crucial for the observability of an infinite set of crossover exponents, not their relevance or irrelevance. In the multifractal descriptions of disordered systems, there seems indeed to be an additional normalization freedom (e.g. scaling at constant voltage or constant current) which allows one to arbitrarily shift the crossover exponents (while maintaining the observable quantities unchanged). The latter arbitrariness seems to be a real difference between multifractals on the one hand, and multicritical behavior near a transition with a continuous symmetry on the other. Also, as pointed out by DUPLANTIER [21], a purely geometrical interpretation (such as that of Ref. [5]) of the Legendre transform of the ϕ_n does not necessarily hold. Nevertheless, the analogies are in fact so close that they have suggested, for multifractals, whole classes of universal quantities which we have calculated ([7]-[9]) but which have yet to be measured.

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