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a.c. response of fractal networks (*)

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Résumé. — Nous avons calculé la réponse en fréquence en courant alternatif pour un réseau du type Sierpinski dans lequel les liens sont soit des résistances R (ou des impédances Z_b) et où tous les nœuds sont reliés à la terre par l'intermédiaire de capacités C identiques (ou d'impédances Z_v). Pour toutes les fréquences plus petites que $1/RC$, l'admittance complexe résultante entre chacun des nœuds « principaux » et la terre peut être exprimée avec précision au moyen d'une fonction d'échelle avec effet de taille finie, tous les exposants de cette fonction étant des combinaisons des dimensions fractale d_f et spectrale d_s du tamis de Sierpinski. La dépendance en fréquence de la fonction de réponse présente une très forte ressemblance avec celle d'un mélange aléatoire de particules conductrices et isolantes.

Abstract. — We calculate the a.c. frequency response of Sierpinski-gasket networks, in which the bonds consist of resistors R (or of impedances Z_b) and all nodes are connected to the circuit ground by identical capacitors C (or by impedances Z_v). The resulting complex, size-dependent admittance between any of the « principal » nodes and the circuit ground can be accurately described at all frequencies less than $1/RC$ by a finite-size scaling function whose exponents are combinations of the fractal dimension d_f and the spectral or « fracton » dimension d_s of the Sierpinski gasket. The response function also bears a striking similarity to experimental observations of the a.c. response of a random mixture of conducting and insulating particles.

(*) Preliminary results of this paper were presented in poster form at the Third Conference on Fractals, « Fractals in the Physical Sciences », NBS, Gaithersburg, Maryland, November 21-23, 1983.

1. Introduction.

It is generally recognized that a fundamental property of percolation clusters is their self-similarity. This has prompted Gefen *et al.* [1] to suggest that self-similar fractal lattices such as the Sierpinski gasket might be a useful tool to understand percolation since the properties of these lattices can in general be computed exactly. This idea has been explored further. Alexander and Orbach [2] and Rammal and Toulouse [3] inspired by the properties of the Sierpinski gasket, have derived interesting scaling laws which apparently apply to real percolation clusters. One should keep in mind that, as pointed out before deterministic *nonhomogeneous* fractals, such as Sierpinski gaskets, display only qualitative, not quantitative, similarities with real percolation clusters.

The model we consider in this paper is the Sierpinski gasket (Fig. 1) with nodes interconnected through impedances Z_h (we specifically treat the case for which Z_h is purely resistive), in which each of the circuit nodes is connected to the circuit ground through an impedance Z_v (which here we treat as purely capacitive). This model is admittedly somewhat artificial, but has the advantage that it is exactly soluble, and that it can be proved that a scaling function exists from which properties of the system can be computed.

The scaling function we propose accurately describes the frequency-dependent admittance $Y(i\omega)$ for a Sierpinski gasket with $Z_h = R$, $Z_v = 1/i\omega C$, throughout the frequency range $\omega < 1/RC$. We, in fact, also show that for this network the scaling function can be explicitly *derived*. We further find a low-frequency crossover which is not only size-dependent but also in qualitative agreement with some experimental data obtained on heterogeneous metal-dielectric mixtures. We find an additional crossover at $\omega > (RC)^{-1}$ which also seems to be in good agreement with some experimental data on the loss tangent.

We proceed to show how the admittance function $Y(i\omega)$ can be exactly calculated by circuit-theory methods, developing recursion relations for the various iterations of the Sierpinski gasket. We give physical arguments for the scaling-function analysis and show that the exactly calculated frequency and size dependence of $Y(i\omega)$ accurately follows the power laws predicted on the basis of the scaling function. Finally, we point out in what sense some of our results are similar to those obtained for real systems, even though quantitative agreement would be expected only for systems with grounded capacitors at each node. (While we treat in detail the case of the two-dimensional Sierpinski gasket, which is evidently the easiest to visualize and to illustrate, we will equally quote the results for the general d-dimensional gasket. We have actually carried out those calculations as well, and find that they follow the corresponding predictions just as well as the two-dimensional case detailed in this paper.)

In the appendix, we explicitly derive the scaling function from a generating-function approach, and we carry out the fixed-point analysis. The admittance we study there would be the analog of the inverse of a single-site density-of-states in the mass and spring problem [4] whose electrical analog has Z_h inductive instead of resistive.

2. Exact network solution and recursion relations for $Y(i\omega)$.

Figure 1 provides a reminder of the iterative procedure for constructing the Sierpinski gasket. The network is built up by starting out with the basic « element » sketched in figure 1a for the *RC* network, and with that of figure 1b for the general case. Thus the bonds of the Sierpinski gasket become resistors R (impedances Z_h, in general), while each node of the basic « element » is connected to circuit ground through a capacitance $C/2$ (in the general case, through an impedance $2\,Z_v$). When the blocks are connected to construct the next iteration of the Sierpinski gasket, this results in a capacitance C (impedance Z_v) to ground from each internal node of the resultant network, and $C/2$ (impedance $2\,Z_v$) from the *principal* nodes at the corners. In the ultimate calculation of the admittance $Y(i\omega)$ at a principal node, two gaskets are connected together as illustrated in figure 1c, thus giving an identical capacitance C to ground from every electrically

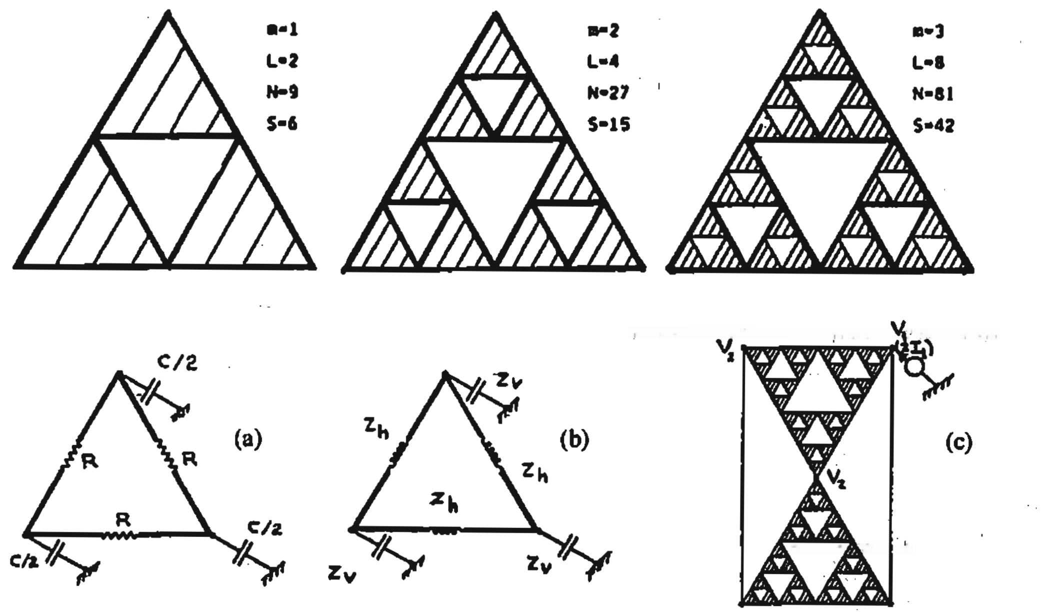


Fig. 1. — The two-dimensional Sierpinski-gasket circuits. The upper three figures illustrate the traditional iterative construction. Each of the shaded areas represents the basic circuit shown in detail in the figures (a) or (b) below (m : iteration ; L : linear dimension of a side ; N : the number of bonds in the lattice ; S : the number of sites, or nodes, of the lattice). (c) illustrates the exact electrical configuration for which the admittance is calculated at the principal node where the power source is shown connected.

independent node. (In the case of Sierpinski gaskets of higher dimensionality d, the procedure is a generalization to d-dimensions of the two-dimensional triangle and of the three-dimensional regular tetrahedron which are used to generate the Sierpinski gasket in the corresponding embedding space.)

For the exact calculation, each individual « element » of the gasket is considered a three-terminal network for which the voltages at the *principal* nodes are related to the three currents entering the corresponding nodes by a (3×3) *z-matrix*. By symmetry, all diagonal elements of the matrix have an equal value z, while all off-diagonal elements have the value z_1. Operationally, these quantities are defined by the relations $z = V_1/I_1$, $z_1 = V_2/I_1$, where V_1 is the voltage at principal node 1 when a current I_1 is injected into that node, and the currents entering *all other nodes* are zero, while V_2 is the voltage at any other principal node under the same conditions. With the construction of figure 1c, the admittance between node 1 and ground is then very simply $Y(i\omega) = 2/z$. Because of the iterative construction of each Sierpinski gasket from the previous one, the impedance coefficients themselves can be shown to obey the recursion relations [1] :

$$z_1' = \frac{z_1^2}{2\,z + z_1} \quad \text{and} \quad z' = z - 2\,z_1' \tag{1}$$

[1] For the d-dimensional gasket :

$$z_1' = z_1^2/\{\,2\,z + z_1(d-1)\,\} \quad \text{and} \quad z' = z - dz_1'$$

with initial conditions for the RC case :

$$z_1 = 4/\{\,(i\omega C)\,(2(d+1) + i\omega\tau)\,\} \quad \text{and} \quad z = z_1(1 + i\omega\tau/2)\,.$$

where the primed quantities are those corresponding to the newly iterated gasket, and the unprimed ones refer to the previous iteration. The initial values corresponding to the iteration $m = 0$ of the « fundamental » building block (Fig. 1a) are :

$$z_1 = 4/\{(i\omega C)(6 + i\omega\tau)\} \quad \text{and} \quad z = z_1(1 + i\omega\tau/2) \tag{1a}$$

for the RC case with $\tau = RC$, while for the LC case we have :

$$z_1 = 4/\{(i\omega C)(6 - \omega^2 LC)\} \quad \text{and} \quad z = z_1(1 - \omega^2 LC/2).$$

We can therefore calculate z and hence $Y(i\omega) = G(\omega) + i\omega C(\omega)$ as functions of frequency, obtaining the results displayed in figures 2, 3, and 4, which show respectively the loss tangent $\{\tan\theta = \omega C(\omega)/G(\omega)\}$, $C(\omega)$, and $G(\omega)/\omega$ as functions of the frequency.

3. Scaling predictions.

We now present the major results and physical consequences of the scaling approach. (For a detailed treatment and fixed-point analysis, the reader is referred to the appendix.)

As usual in applications of finite-size scaling, we write the admittance as a generalized homogeneous function

$$Y(i\omega\,; L) = L^{\alpha}\, F(i\omega\tau_{\rm D}\,; i\omega\tau) \tag{2}$$

where L is the linear dimension of the system, in units of the lattice spacing, $\tau = RC$ as before, α is an exponent defined below, and $\tau_{\rm D}$ is a characteristic diffusion time for the given lattice, which may be obtained from detailed random-walk considerations [5]. The scaling of $\tau_{\rm D}$ with L can also be obtained from a very simple physical argument : Rammal and Toulouse [3] have indicated that the mean number of distinct sites visited during a random-walk of t steps varies as $t^{d_s/2}$, where d_s is the spectral or « fracton » dimensionality. In any fractal lattice, the number of sites, or bonds, varies as L^{d_f}, with d_f the fractal dimension. Equating these two numbers, we obtain that $\tau_{\rm D}$, the characteristic time to explore given finite lattice of size L scales as L^{d_w}, with the random-walk exponent $d_w = 2\, d_f/d_s$. Thus, the characteristic scaling variable of equation (2), $(i\omega\tau_{\rm D})$, can also be written as $i\omega L^{d_w}$. The latter result and a scaling function analogous to (2) were suggested before by Straley [6]. Note that for a real percolation problem, one expects [5] $i\omega L^{(s+t)/\nu}$ as a scaling variable instead of $i\omega L^{d_w}$. As shown in the appendix, the scaling variables $i\omega L^{d_w}$ is simply related to the ratio z/z_1 of the quantities defined in equation (1) above.

Equation (2) describes in fact three different regions, with crossovers between them at $\omega\tau_{\rm D}$ $(= \omega L^{d_w}) \sim 1$ and $\omega\tau \sim 1$ respectively. Since $\tau_{\rm D}$ increases with L, we have $\tau_{\rm D} \gg \tau$ for large L, so that starting from low frequencies, we can at first set $\omega\tau \sim 0$, and we have for the first two regions :

$$Y(i\omega\,; L) = L^{\alpha}\, F(i\omega L^{d_w}\,; 0)\,. \tag{2a}$$

With the usual finite-size scaling assumption that F is an analytic function, we can expand (2a) at very low frequencies in a power series obtaining

$$Y(i\omega\,; L) = G(\omega) + i\omega C(\omega) = L^{\alpha}\,\{F(0\,;0) + i\omega L^{d_w}\,F'(0\,;0) - \omega^2\,L^{2d_w}\,F''(0\,;0) + \cdots\}\,. \tag{3}$$

Since $G(0)$ must vanish, as there is physically no resistive path to ground in the circuit, we have $F(0\,;0) = 0$ and we thus find that for low frequencies $C(\omega) = L^{\alpha+d_w}\,F'(0\,;0)$. Since further, at very low frequencies, resistive bonds have negligible potential drops across them, all capacitors add in parallel. Thus the total capacitance, proportional to the number of sites, varies as L^{d_f}, giving us $\alpha = d_f - d_w$. We therefore further predict the low frequency behaviour of $G(\omega) \sim \omega^2 L^{d_f + d_w}$ and of the loss-angle tangent $\tan\theta \sim 1/\omega L^{d_w}$. For Sierpinski gaskets in d-dimensions, the exponents are exactly known [3] : $d_f = \ln(d+1)/\ln 2$, and $d_s = 2\ln(d+1)/\ln(d+3)$, (giving $d_w = \ln(d+3)/\ln 2$). We have been able to verify these predictions both with calculations on electrical networks and on generating functions (Appendix). Note that the result $\alpha = d_f - d_w = d_f\left(1 - \frac{2}{d_s}\right)$ was found in reference [3] (where it was called β_L) for geometries where the conductance is finite at $\omega = 0$. In our case $G \sim \omega^2 L^{\beta_L + 2d_w}$ instead of $G \sim L^{\beta_L}$.

The scaling expression (Eq. (2a)) predicts further a first crossover at $\omega L^{d_w} \sim 1$ and the behaviour in the second frequency region, beyond this first crossover. Indeed, in that region, the behaviour of $Y(i\omega; L)$ must become independent of L

$$Y(i\omega; \infty) = L^{d_f - d_w} F(i\omega L^{d_w}; 0) \tag{4}$$

which can hold provided that $F(x; 0) \sim x^u$ for large x, with $u = 1 - d_s/2$. Then

$$Y(i\omega; \infty) \sim (i\omega)^{1 - d_s/2} = (\omega\, e^{i\pi/2})^{1 - d_s/2} \tag{5}$$

whence also

$$\tan\theta = \mathrm{Im}\, Y/\mathrm{Re}\, Y = \tan\left\{\frac{\pi}{2}\left(1 - \frac{d_s}{2}\right)\right\}. \tag{6}$$

We thus find a frequency region in which the loss-angle tangent is frequency independent, with a predicted value having *no adjustable parameter*. This is verified in the exact results displayed in figure 2. Equations (4) to (6) are discussed in greater detail in the appendix.

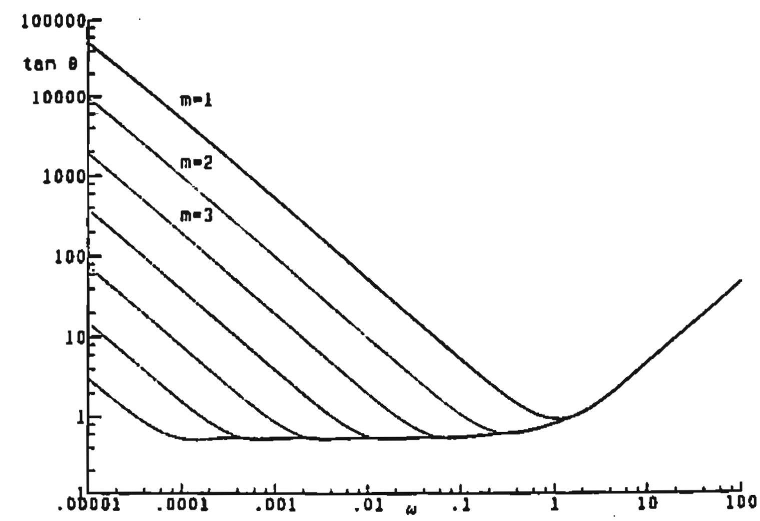
Finally, the last crossover point is seen, from equation (2), to occur at $\omega\tau$ ($= \omega RC$) ~ 1. This follows almost trivially from the physical fact that for $\omega\tau \gg 1$ all capacitors become virtual short-circuits to ground. By inspection of figure 1c, we thus conclude that $Y(i\omega)$ must then be the parallel combinaison of a single capacitance at the measuring node, in parallel with four resistors (for the two-dimensional gasket). All dimension-dependent effects then disappear and $G(\omega)$ and $C(\omega)$ become constants, with the loss tangent proportional to ω.

In summary then, the loss tangent should display three distinct regions : one at low-frequencies, size-dependent, and varying as $1/\omega$, one constant at intermediate frequencies, and one at high frequencies varying as ω. This is seen explicitly in figure 2, with $C(\omega)$ and $G(\omega)$, displayed in figures 3, 4, also following quite accurately the above predictions.

Finally, let us consider in figure 5 experimental data [7] on the loss tangent of a percolation system near but below threshold. A $1/\omega$ behaviour is clearly seen for at least two decades at low frequencies and a ω behaviour at high frequencies for about one decade. These two regimes are clearly in agreement with our results. Furthermore the experimental data depend on (p-p_c) at low frequencies and not at high frequencies. This is also consistent with the fact that our results are size dependent at low frequencies and size independent at high frequencies. While we predict a constant intermediate regime, this is not yet clearly confirmed by experiment. More data are needed in that region.

The experimental system is expected to have a fractal structure at length scales smaller than the correlation length but it certainly does not have grounded capacitors at each node. The above agreement between theory and experiment is thus at first surprising. All this means however, is that the equivalent high and low frequency circuits of the model and of the real system have something in common. In the Sierpinski case, the high frequency equivalent circuit reduces to a capacitor in parallel with a resistor. The result is size independent because the capacitors short the system to ground very rapidly. At low frequencies the equivalent circuit is a size dependent (renor-

(a)



(b)

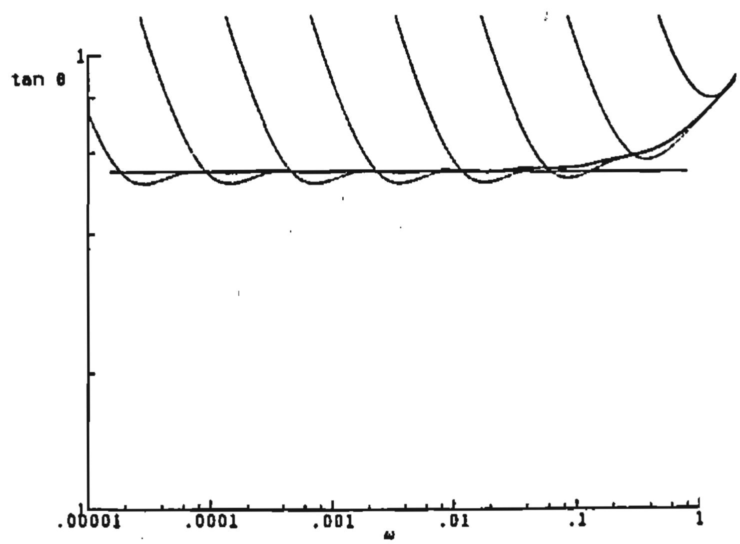


Fig. 2. — (a) Loss tangent *vs.* frequency (measured in units of $1/\tau$). We observe here the unit slopes in both the low- and high-frequency domains, the characteristic minimum region where the loss is independent of both frequency and sample size, and the low-frequency crossover breakpoint which shifts to lower and lower frequencies as the system size increases ($\tan\theta = \omega C/G$). (b) Expanded detail of figure 2a. The horizontal line is the predicted asymptotic mid-frequency value $\tan\theta = \tan[(1 - (d_s/2))(\pi/2)] = 0.5444$.

malized) resistor in series with a size dependent capacitor. More details on that may be found in the appendix.

In the real system, since $p < p_c$, there exists a continuous path of dielectric from one electrode to the other which one may represent by a capacitor C_1. Elsewhere, where there are resistors and capacitors in series, the resistors dominate the admittance at high frequencies and they are in

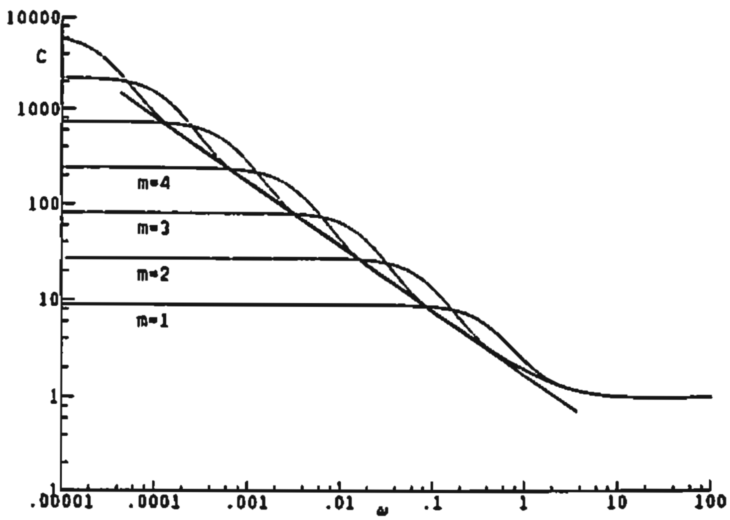


Fig. 3. — Variation of the effective capacitance with frequency. The low frequency plateaus correspond to the sum of all capacitances for the given network, and coincide exactly with the theoretically expected value. The straight line is drawn with the theoretical slope $(-\,d_s/2) = -\,0.6826$.

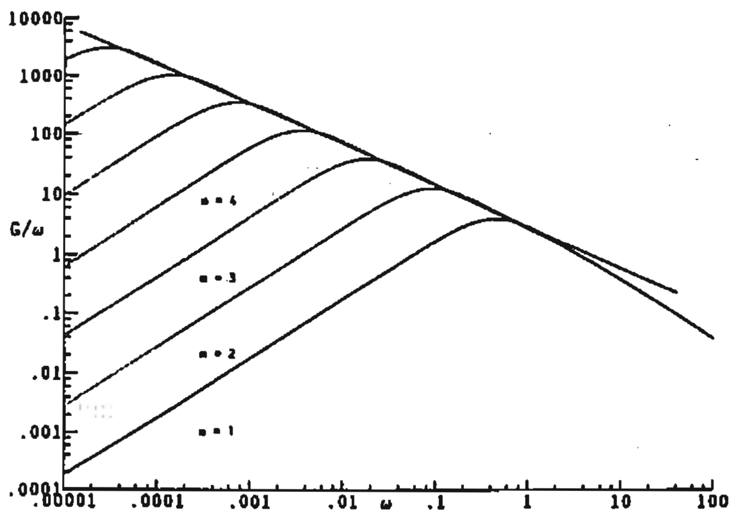


Fig. 4. — Real part of the admittance (displayed as G/ω) *vs.* frequency. Here again the straight line tangent is drawn with slope $(-\,d_s/2)$.

parallel with C_1, which means that $\tan\theta \sim \omega$. This is the same equivalent circuit as the Sierpinski gasket. The admittance is weakly dependent on p-p_c because : *a*) it is the most direct dielectric conduction path which dominates the value of C_1, and this dielectric path still exists for $p < p_c$ in three dimensions; *b*) at high frequencies, the total resistance does not depend very much on whether or not there exists an infinite cluster of resistors since anyway the capacitors act as shorts. At low frequencies on the other hand one must first renormalize up to length scales

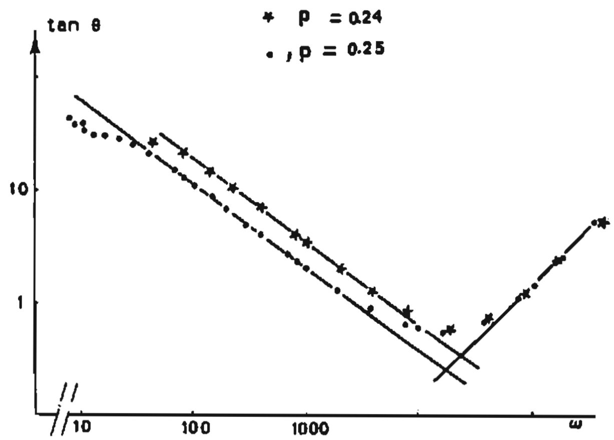


Fig. 5. — Experimental curve of loss tangent *vs.* frequency. These data were obtained on a mixture of silvered and unsilvered glass microbeads just below the percolation threshold. (Courtesy of Clerc, Giraud and Laugier.)

of the order of the correlation length to find effective circuit elements such as C_1. The value of these effective admittances is thus going to depend on p-p_c. When effective resistors are in series with effective capacitors, the capacitors dominate. A power series expansion leads to admittances of the form, $i\omega C + \omega^2 R$. The effective parallel capacitor C_1 simply adds to C in the latter expression. Once more this is the same low frequency equivalent circuit as for the Sierpinski gasket.

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Appendix.

For definiteness, we restrict ourselves to the two-dimensional Sierpinski Gasket (S.G.) illustrated in figure 1. We use periodic boundary conditions. The nodes of the usual lattice are connected with each other by a complex impedance Z_h and to the ground directly through an impedance Z_v. Setting the sum of the currents at each node equal to zero and cancelling the sum of the voltages around each loop formed by an impedance Z_h and the two impedances Z_v connected to it, one finds the following equations of motion for the currents I_i going to the ground through the impedance Z_v at node i,

$$\left(-\frac{Z_h}{Z_v} - 4\right) I_i + \sum_j I_{i+j} = -\frac{Z_h}{Z_v} I_{\text{ext}}^i . \tag{A.1}$$

I^i_{ext} is an external current fed at node i and the sum over j is over the four ($= 2\,d$) nearest neighbours to the node i. If Z_h is inductive ($= i\omega L$ with ω the circular frequency and L the inductance) and Z_v capacitive ($= (i\omega C)^{-1}$ with C the capacitance) then $Z_h/Z_v = -\omega^2 LC$. In this case, clearly, equation (1) is the exact electrical analog of the mass and spring problem considered by various authors [2-4, 8]. In our problem, Z_h is resistive and Z_v capacitive, i.e. $Z_h/Z_v = i\omega RC$.

To study the scaling properties of the admittance $Y_{ij} = I^j_{\text{ext}}/V_i$, where $V_i = Z_v\, I_i$ is the voltage measured between node i and the ground, we use a generating function for the equations of motion (A.1) which is the analog of the free energy in critical phenomena. Calculations of impedances for resistor-capacitor networks using a generating function have been performed before by Stephen [9]. The generating function is obtained as follows [9-11]. Let us first write equation (A.1) for all nodes in the form,

$$\sum_j \left(-\frac{Z_h}{Z_v} \mathbb{J} - H \right)_{ij} I_j = -\frac{Z_h}{Z_v} I^n_{\text{ext}}\, \delta_{in} \tag{A.2}$$

where H is a real symmetric matrix and $\mathbb{J}$ is the identity matrix. Then let U^{T} be the row vector

$$U^{\text{T}} = (u_1, u_2, \ldots, u_N) \tag{A.3}$$

where the u_i are continuous variables which are defined for every one of the N lattice sites i

$$\phi^{\text{T}} = (\phi_1, \phi_2, \ldots, \phi_N), \tag{A.4}$$

then the generating function is, in matrix notation

$$F = \ln \left\{ \prod_{i=1}^{N} \int \mathrm{d}u_i \exp\left[-\frac{i}{2} U^{\text{T}} \left(-\frac{Z_h}{Z_v} \mathbb{J} - H \right) U + \phi^{\text{T}}\, U \right] \right\}. \tag{A.5}$$

The impedance is the analog of the Green's function, hence the admittance may be obtained from,

$$Y_{ij} = \left(-Z_h\, i \frac{\partial^2 F}{\partial \phi_i\, \partial \phi_j} \right)^{-1}_{\phi = 0}. \tag{A.6}$$

From now on, we work in units where R and C are unity, hence $Z_h/Z_v = i\omega$. Note that the argument of the exponential in (A.6) is such that the integrals are convergent.

We are interested in Y_{11}, the admittance measured at one of the principal nodes in figure 1. This means that we can use equation (A.5) with all the elements of the row vector (A.4) equal to zero except that one, ϕ, which corresponds to the site of interest. Then one can compute,

$$Y_{11} = \left(-i \frac{\partial^2 F}{\partial \phi^2} \right)^{-1}_{\phi = 0}. \tag{A.7}$$

F may be obtained from an exact renormalization group transformation [12] : integrals over the u_i on triangles at the smallest scale are performed and renormalized parameters are defined so that the generating function is preserved [11]. The recursion relations for the diagonal and off-diagonal elements of H may be found in the paper by Domany *et al.* [8]. The so-called « constant term » in F, which is independent of the non-integrated variables, is also clearly independent of ϕ since u_1 is not integrated over. This means the « constant term » will not contribute to Y_{11} (Eq. (A.7)) and hence may be dropped. One then rescales the non-integrated variables so that the off-diagonal elements of H become equal to unity as they were at the beginning. This introduces (through the Jacobian) another ϕ independent term in H' which may be dropped. One is left with two recursion formulas, one for the diagonal element of $(-i\omega 3 - H)$, say ε, and one for ϕ. The recursion formula for the variable $x = (\varepsilon + 4)/5$ takes the well known form

$$x' = \lambda x(1 - x) \qquad (A.8)$$

with $\lambda = 5$. This map for x in the complex plane, the case of interest to us, has been discussed by Mandelbrot [13]. Since the initial value of x is $x = -i\omega/5$, it is more convenient to work in terms of a variable ω' which obeys

$$\omega' = \omega(5 + i\omega) . \qquad (A.9)$$

In terms of this variable, the recursion formula for ϕ is,

$$\phi' = \phi\left[\frac{(2 + i\omega)(5 + i\omega)}{6 + i\omega}\right]^{1/2} . \qquad (A.10)$$

The above arguments and the invariance of F under renormalization imply

$$\begin{aligned} Y_{11}(i\omega; L) &= \left[-i\frac{\partial^2 F(i\omega;\phi;L)}{\partial\phi^2}\right]^{-1}_{\phi=0} \\ &= \left[-i\frac{\partial^2 F(i\omega';\phi';L/b)}{(\partial\phi')^2}\right]^{-1}_{\phi=0}\left(\frac{\partial\phi}{\partial\phi'}\right)^2 \end{aligned} \qquad (A.11)$$

where we have explicitly written down the size dependence, L, in units of the lattice spacing and defined $b = 2$ as the length rescaling factor. Equation (A.11) becomes our basic result :

$$Y_{11}(i\omega; L) = \left(\frac{\partial\phi}{\partial\phi'}\right)^2 Y_{11}(i\omega'; L/b) . \qquad (A.12)$$

To understand the scaling behaviour of (A.12), one thus needs to understand the fixed points of equation (A.9). These are located at $\omega = 0$, $\omega = 4\,i$ and $|\omega| = \infty$. The fixed point at $\omega = 0$ is unstable in the real and imaginary directions with an eigenvalue 5 while $|\omega| = \infty$ is stable. These two fixed points are those which interest us. We will not discuss the $\omega = 4\,i$ fixed point because it is unstable in all directions and there is no initial value of ω for our problem which is close to that fixed point. There are three typical frequency regions for Y_{11}.

a) $\omega L^{d_w} \ll 1$.

Close to the fixed point $\omega = 0$, the recursion relation (A.9) may be linearized and equation (A.12) reduces to

$$Y_{11}(i\omega; L) = b^{d_f - d_w}\, Y_{11}(ib^{d_w}\omega; L/b) \qquad (A.13)$$

where we defined $d_f = \ln(3)/\ln 2$ and $d_w = \ln(5)/\ln 2$, with $b = 2$. d_f is the usual fractal dimension and d_w is related to the anomalous diffusion exponent [2, 3, 5]. When condition a) is satisfied, the linearization of the recursion formula remains valid even if one iterates n times, up to $L/b^n = 1$. Then $Y_{11}(ib^{nd_w}\omega; 1)$ is the impedance at frequency ωL^{d_w} of a S.G. formed of a single unit (with periodic boundary conditions) which may easily be calculated as, $Y_{11}(ix; 1) = (6\,ix - x^2)/(2 + ix)$. Thus, when a) is satisfied,

$$Y_{11}(i\omega; L) \sim L^{d_f - d_w}((L^{d_w}\omega)^2 + 3\,i(L^{d_w}\omega)) . \qquad (A.14)$$

which agrees with equation (3).

b) $L^{-d_w} < \omega < 1$.

It is clear from the recursion relation (A.9) that non-linear terms start to be important when $\omega \sim 5$. When $\omega < 1$, the recursion relation may be linearized and equation (A.13) still applies. Since in b) we have also $L^{-d_w} < \omega$, we can iterate up to $b^{nd_w}\,\omega \sim 1$ while remaining in the limit $(L/b^n) \sim (L\omega^{1/d_w}) \gg 1$. When $L' \gg 1$, $Y_{11}(i; L')$ is independent of the system size because at those frequencies ($\omega \sim 1 \sim 1/RC$) the capacitors are becoming short-circuits and do not allow the input current to go very far down the S.G. If equation (A.13) is iterated n times then

$$Y_{11}(i\omega; \infty) = b^{n(d_f - d_w)}\, Y_{11}(ib^{nd_w}\,\omega; \infty). \tag{A.15}$$

Equation (A.15) must be independent of n. This will be so if [14]

$$Y_{11}(i\omega; \infty) \approx b^{n(d_f - d_w)}(ib^{nd_w}\,\omega)^u\, f(t)$$
$$t = \log(i\omega b^{nd_w})/\log(b^{d_w}) \tag{A.16}$$

with $u = (d_w - d_f)/d_w = 1 - d_s/2$ and $f(t + 1) = f(t)$ a periodic function. The admittance must be a smoothly varying function of ω because the eigenvalues of H in equation (A.2) are real and finite while $-Z_h/Z_v = -i\omega$ is imaginary which means that there are no poles nor zeros in the impedance matrix for ω real (by contrast with what happens in the LC case). It is thus a good approximation to take f as independent of ω. In that case f must be real since the imaginary quantity i appears only in the combination $i\omega$ which means that a function which is independent of ω must also be real. Hence we are left with

$$Y_{11}(i\omega; \infty) \sim (i\omega)^{1 - d_s/2} \tag{A.17}$$

in agreement with equation (5).

c) $\omega \gg 1$

Then one is clearly very close to the domain of attraction of the fixed point at $|\omega| = \infty$ which we consider as the analog of a phase sink (trivial) fixed point. In that regime, one may compute the admittance of the S.G. from that of the first elementary triangles attached to the current source. The effect of triangles further away may be taken into account perturbatively for example and is of higher order in powers of $1/\omega$. One thus finds,

$$Y_{11}(i\omega; L) \sim 4 + i\omega. \tag{A.18}$$

The above three asymptotic regimes, equations (A.14), (A.16) and (A.18) may be seen explicitly from the results of numerical calculations exhibited in figures 3 and 4.

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