How many correlation lengths for multifractals?

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It is shown, here in the context of percolation, that one can have multifractal behavior and at the same time a single correlation length. This length sets the scale below which non-trivial scaling behavior occurs and it is controlled by a few relevant operators, as in ordinary critical phenomena. All other correlation lengths may be obtained from simple metric factors. The quantities which lead to an infinite set of exponents are described by a second renormalization group which is slaved to the first one which determines the correlation length. The results are relevant also to the problem of noise in percolating systems.

1. Introduction

In the last five or six years, the concept of multifractals has become widely used in various contexts related to fractal, self-similar behavior. Anticipatory work by Mandelbrot had appeared several years before [1]. Although the concept at first seemed extremely different from anything seen before in critical phenomena, it is now apparent that there are in fact many analogies. One question which the first author heard formulated by Michael Fisher in 1986 at Cornell concerns the uniqueness of the correlation length. Multifractal behavior is usually characterized by the presence of an infinite set of observable exponents. Since one is used to only a few relevant operators in critical phenomena [2], it is then natural to ask if this infinity of observable exponents comes about because there is an infinity of correlation lengths! Without trying
to guess what motivated professor Fisher's question, we recognize that it is a key question to understand the origin of multifractal behavior. Indeed, professor Fisher stressed in some of his lectures on critical phenomena at Cornell that "There is one length and this length is the correlation length. This is a key to understand critical phenomena and the renormalization group." The correlation length is thus at the core of our understanding of the renormalization group and hence of the existence of a few relevant exponents.

Several authors have answered the title question without necessarily addressing it explicitly [3-5]. Some have implicitly calculated as if there was a single length [3], others have defined an infinity of correlation lengths without bothering about the deeper meaning of the definition, and others have made statements which imply an infinity of correlation lengths without being moved by the far reaching consequences of their statements [5]. In the present paper in honor of professor Fisher, we answer the title question by "Any correlation length is related to a reference correlation length by a constant which, in the scaling regime, does not depend on the distance to the critical point." This short answer will be made more precise in the rest of the paper. We also have to be careful in defining clearly what we mean by multifractal. There are analogies between multifractals and critical phenomena, and analogies between multifractals and multifractals but not all multifractals are created equal, nor are they completely identical with critical phenomena [6-10]. There are multifractals in percolation [11-14] dynamical systems [15], localization [16], diffusion limited aggregation [17], turbulence [1], ... . We believe that in many respects the structure of the underlying renormalization group description is very similar for the first three cases only. And these three cases show both similarities and differences with critical phenomena. These similarities and differences have been discussed before [6-10]. By focusing on the question "How many correlation lengths?", we once more have to address analogies and differences with critical phenomena but our discussion here focuses on the correlation length, and it thus provides a fresh perspective on earlier results. The reader will be referred to these earlier references for more details [8, 18].

To be specific, we concentrate on the case of percolation but we also want to stress that our perspective also applies to the field of dynamical systems, as discussed elsewhere [19]. We leave aside localization, but we have good reasons to believe that it is a case whose general structure shows many similarities with percolation.

2. Multifractals in percolation

To recall how multifractals arise in percolation, we first look at some general properties concerning the noise of resistance networks [11, 12]. Consider a
resistance network whose individual components are fluctuating around some
mean value. Cohn's theorem shows that, to linear order in the fluctuations
of the individual resistances \( \delta r_\alpha \), the change of the macroscopic resistance \( \delta R \) can
be written as [11, 12]

\[
\delta R I^2 = \sum_\alpha i^2_\alpha \delta r_\alpha ,
\]

(1)

where \( I \) is the total input current and \( i_\alpha \) is the current in bond \( \alpha \) when all the
resistances have their mean value. Expression (1) is reminiscent of linear
response theory and is valid only when \( (\delta r_\alpha / r_\alpha) < 1 \). We only consider models
where each of the resistances is fluctuating independently. The time Fourier
transform of the macroscopic resistance fluctuations can then be written as

\[
\langle \delta R \delta R \rangle_\omega = \sum_\alpha (i^2_\alpha)^2 \langle \delta r_\alpha \delta r_\alpha \rangle_\omega ,
\]

(2)

where \( \langle \delta r_\alpha \delta r_\alpha \rangle_\omega \) is the Fourier transform of the microscopic resistance
fluctuations of the bond \( \alpha \). It is understood from now on that the input current
\( I \) is equal to unity. The problem of the microscopic origin of the noise is not
addressed. For most experimental cases of 1/f noise [20], the hypothesis of
independent resistance fluctuations should be valid [11, 12]. To simplify the
discussion the index \( \omega \) is dropped. Generalizing eq. (2), the \( n \)th cumulant of
the macroscopic resistance fluctuations can be expressed as

\[
\langle \delta R^n \rangle = \sum_\alpha (i^2_\alpha)^n c_n(\alpha) ,
\]

(3)

where \( c_n(\alpha) \) is the \( n \)th cumulant of the microscopic resistance fluctuations of
bond \( \alpha \). The case \( n = 1 \) in eq. (3) can be interpreted as an expression for the
macroscopic resistance.

We now come back to percolation, that is to a two-component random
resistance network where each bond has a resistance \( r_g \) (good conductors) with
probability \( p \) and \( r_b \) (bad conductors) with probability \( 1 - p \). The latter
resistance \( r_b \) is taken as larger than \( r_g \) in all problems we consider. When \( r_g \) is
finite and \( r_b \) infinite, we have the conductor–insulator problem, while when \( r_g \)
is zero and \( r_b \) finite, we have the superconductor–conductor problem. We note
\( c_g(n) \), the cumulants of the microscopic resistance fluctuations of the bonds
having resistance \( r_g \), and we note \( c_b(n) \), the cumulants of the microscopic
resistance fluctuations of the bonds having resistance \( r_b \). For a given realization
of disorder one finds, using eq. (3), that the cumulants of the macroscopic
resistance fluctuations are given by
\[ \langle \delta R^n \rangle = \left[ \sum_{\alpha(g)} (i_n^2)^\alpha \right] c_g(n) + \left[ \sum_{\alpha(h)} (i_n^2)^\alpha \right] c_h(n). \]  

where the first sum is restricted to the bonds having resistance \( r_g \) and the second to the bonds having resistance \( r_h \). It remains to take the average with respect to disorder. Working in a finite system of size \( L \) at the percolation threshold \( p = p_c \), the percolating samples (PS) and the non-percolating samples (NPS) are averaged separately. The reason for this will become clear later. By a percolating sample, we mean a sample with a spanning cluster of bonds having resistance \( r_g \), the smallest of the two.

The quantities appearing in square brackets in eq. (4) are called multifractal moments. \( G \)'s will refer to multifractal moments of the good conductors and \( B \)'s to multifractal moments of bad conductors. More specifically, the multifractal moments of the currents (squared) in the bonds of type \( r_g \) and the multifractal moments of the currents (squared) in the bonds of type \( r_h \), averaged over the PS, are noted \( G_p(n, L) \) and \( B_p(n, L) \), respectively. The corresponding quantities averaged over the NPS are noted \( G_{np}(n, L) \) and \( B_{np}(n, L) \). These quantities are computed for a unit applied current \( I \). The cumulants of the observed noise are then related to the multifractal moments through

\[ \langle \delta R^n \rangle_p = G_p(n, L) c_g(n) + B_p(n, L) c_h(n), \]  

\[ \langle \delta R^n \rangle_{np} = G_{np}(n, L) c_g(n) + B_{np}(n, L) c_h(n). \]  

The time average is implicit and is dropped from now on. Also, \( \langle \cdot \rangle_p \) and \( \langle \cdot \rangle_{np} \) refer to averages over respectively the percolating and the non-percolating realizations of the random network. Note that the multifractal moments are a property of the unfluctuating lattice [11-14]. They are moments of the current distribution averaged over realizations of the percolating network, and they are called multifractal because of their peculiar scaling properties at the percolation threshold \( \Delta p = |p - p_c| = 0 \) and \( h = r_g/r_h = 0 \).

\[ G_p(n, L, \Delta p = 0, h = 0) = \left\langle \sum_{\alpha(g)} (i_n^2)^\alpha \right\rangle_p \sim L^{-x_n}, \]  

\[ B_{np}(n, L, \Delta p = 0, h = 0) = \left\langle \sum_{\alpha(h)} (i_n^2)^\alpha \right\rangle_{np} \sim L^{-x_n}. \]  

At the most simple-minded level, we say we have multifractal behavior because we have the infinite sets of exponents \( -x_n \) and \( y_n \) and they do not obey the familiar linear dependence on \( n \) encountered in the case of gap scaling [2].
Note that in the superconductor–conductor case, it is more natural to work with voltages instead of currents. In this case, it is the equivalent set of exponents \( z_n = 2ny_i - y_n \) which arises more naturally [18, 21]. We will nevertheless stick to the \( y_n \) to keep the discussion as straightforward as possible.

3. An infinite number of correlation lengths?

The following discussion is for \( \Delta p = 0 \), \( L \) finite, but it can also be transposed to the case \( \Delta p \) finite, \( L \to \infty \). Let us first recall how we obtain the correlation length at \( h \) finite. We know that in percolating samples,

\[
\langle R \rangle_p = r_\phi L^{-1} g_p(hL^\phi),
\]

while in non-percolating ones,

\[
\langle R \rangle_{\text{np}} = r_\phi L^{-1} \beta_{\text{np}}(hL^\phi).
\]

When \( hL^\phi \gg 1 \), the resistances in eqs. (9) and (10) should scale as \( L^{2-d} \) and should give identical results. The appropriate power-law dependence on \( L \) can be obtained if the two crossover functions \( g_p(u) \) and \( \beta_{\text{np}}(u) \) have power-law behavior \( u^{\alpha_1+2-d+\phi} \) and \( u^{\alpha_2+2-d+\phi^*} \), respectively, while the equality of the two resistances in eqs. (9) and (10) follows with \( \phi = -x_i - y_i \). (If we had taken different crossover exponents \( \phi \) and \( \phi^* \) in eqs. (9) and (10), analyticity as a function of \( L^{-1} \) would have implied their equality. This type of argument is usually presented as analyticity as a function of \( \Delta p \) at \( h \) finite in critical phenomena.)

Let us now define the correlation length through a multifractal moment. First, note that the work of several authors on general multifractals [22] suggests that for random resistor networks of size \( L \) at \( p = p_c \) we have

\[
\langle i_\phi^{2n}(r) i_\phi^{2n}(0) \rangle \sim L^n r^{-\nu},
\]

when \( 1 \ll r \ll L \). The exponents \( u \) and \( \nu \) need not be specified any further. We can use such a multifractal correlation function to define the correlation length when \( p \neq p_c \), \( h \neq 0 \),

\[
\xi_n^2 = \frac{\int d^4r \int \left( i_\phi^{2n}(r) i_\phi^{2n}(0) \right)^2}{\int d^4r \left( i_\phi^{2n}(r) i_\phi^{2n}(0) \right)}.
\]

Denoting \( \langle i_\phi^{2n}(r) i_\phi^{2n}(0) \rangle \) by \( \xi_n(r, L, \Delta p, h) \) in the general case and supposing
that, like the corresponding multifractal moment $G_p$, $C_n$ obeys an homogeneity relation where $\Delta p$ is scaled by $\lambda^{-\nu}$ and $h$ by $\lambda^{-\phi}$, we find that

$$\xi_n^2 = \frac{\int \, \text{d}^d r \, r^2 C_n(r/L, L/\lambda, \Delta p \lambda^{1/v}, h \lambda^\phi)}{\int \, \text{d}^d r \, C_n(r/L, L/\lambda, \Delta p \lambda^{1/v}, h \lambda^\phi)} = \lambda^{-v} F_n(L/\lambda, \Delta p \lambda^{1/v}, h \lambda^\phi), \quad (13)$$

where the overall power of $\lambda$ multiplying $C_n$ cancels and where the last equality is obtained by a simple change of integration variables. When $L \to \infty$ and $h = 0$, one chooses $\lambda = \Delta p^{-\nu}$ so that $\xi = a_n \Delta p^{-\nu}$, with $a_n$ a constant. When $L \to \infty$ and $\Delta p = 0$, one chooses $\lambda = h^{-1/\phi}$ so that $\xi = b_n h^{-1/\phi}$, with $b_n$ a constant. In general then, in the infinite size limit where $\xi$ cannot depend on $L$, one can write

$$\xi_n = \Delta p^{-\nu} F_n(h \Delta p^{-\nu}), \quad (14)$$

where $F_n(x) = a_n$ for $x \to 0$, and $F_n(x) = b_n x^{-1/\phi}$ for $x \to \infty$. The constant prefactors $a_n$ and $b_n$ depend on the multifractal correlation function considered in eq. (12). Nevertheless, one usually says in this context that there is a single correlation length, since all lengths are proportional to a single one, independently of the value of $\Delta p$ and $h$: On the usual scaling plots $\ln(\xi/\Delta p^{-\nu})$ vs. $\ln(h \Delta p^{-\nu})$, the constants $a_n$ and $b_n$ can be scaled away by translations. They are called metric factors. In the crossover region, where $h \Delta p^{-\nu}$ is neither small nor large, it takes the full functions $F_n$ in eq. (14) to specify the various lengths. A similar situation arises in critical phenomena if we try to define the correlation function by either the order parameter correlation function or by the energy correlations$^{a1}$ [23]. We thus consider that as long as the dependence on $\Delta p$ and $h$ of all multifractal moments can be scaled like $\Delta p \lambda^{1/v}$ and $h \lambda^\phi$ with exponents $\nu$ and $\phi$ which do not depend on $n$, there is a single correlation length in the problem.

Suppose now that we do as in ref. [5] and, in analogy with eqs. (9) and (10), use the argument that at finite $h$ the multifractal moments for percolating and non-percolating samples are equal. Then, equating the following two finite $h$ equations, which are natural generalizations of the $h = 0$ cases in eqs. (7) and (8),

$$\langle \delta R_n'' \rangle_p = r_p^n L^{-n_\delta} \Phi_p(n, h L^\delta), \quad (15)$$

$$\langle \delta R_n'' \rangle_{NP} = r_p^n L^{-n_\delta} \Phi_{NP}(n, h L^\delta), \quad (16)$$

$^{a1}$ We thank M.E. Fisher for correspondence on this point. Note, for example, that in the Gaussian model, one can easily check that the energy correlation length above $T_c$ is half the field correlation length.
we obtain an infinity of crossover exponents \( \phi_n \) which depend on \(-x_n\) and \(y_n\). In other words, for every multifractal moment, the dependence on \( h \) will need to be scaled like \( h^{\lambda \phi} \). Pursuing the same line of argument which leads to the scaling relation (14) for the correlation length, one obtains this time

\[
\xi_n = \Delta p^{-\nu} \mathcal{F}_n(h \Delta p^{-1/\xi_n}).
\] (17)

When \( \Delta p = 0 \) for example, we have \( \xi_n = b_n h^{-1/\xi_n} \) and because of the \( n \) dependence of the exponent, there is indeed an infinity of correlation lengths: For each value of \( h \), except \( h = 0 \), there are many lengths and their ratio is not independent of \( h \), contrarily to the case where there is a single crossover exponent.

The paradox exposed in this section then, is that the homogeneity relations eqs. (15) and (16) with analyticity of \( \langle \delta R'' \rangle \) at large \( h \) appear inconsistent with the existence of a single crossover exponent (or correlation length).

4. Only one correlation length, but confluent singularities for \( n \geq 2 \)

It is possible then to have a single crossover exponent when we have multifractal behavior? The answer is yes, and it is because instead of the previous homogeneity relations (15) and (16), the multifractal moments obey, near percolation, relations of the type

\[
\langle \delta R'' \rangle_p = c_6(n) \left( \sum_{a \neq 1} \langle i_{a}^{2} \rangle \right)^{n} + c_8(n) \left( \sum_{a \neq 1} \langle i_{a}^{2} \rangle \right)^{\nu} + c_8(n) \lambda^{-x_{n}} G_{p}(n, L/\lambda, \Delta p^{1/\nu}, h \lambda^2) + c_8(n) \lambda^{-x_{n}} \lambda^{\nu} \langle h \lambda^{\phi} \rangle^{2n} \tilde{B}_p(n, L/\lambda, \Delta p^{1/\nu}, h \lambda^2)
\] (18)

for the percolating samples, and

\[
\langle \delta R'' \rangle_{NP} = c_6(n) \ G_{NP}(n, L, \Delta p, h) + c_8(n) \ B_{NP}(n, L, \Delta p, h)
\]

\[
= c_6(n) \lambda^{-x_{n}} \lambda^{\nu} \lambda^{\phi} \langle h \lambda^{\phi} \rangle^{2n} \tilde{G}_{NP}(n, L/\lambda, \Delta p^{1/\nu}, h \lambda^2) + c_8(n) \lambda^{-x_{n}} \lambda^{\nu} B_{NP}(n, L/\lambda, \Delta p^{1/\nu}, h \lambda^2),
\] (19)

for non-percolating ones [18, 24]. These homogeneity relations have been suggested by a simple Migdall–Kadanoff renormalization group, and verified by
Monte Carlo simulations [18]. The four scaling functions in (18) and (19) have been defined so that they become constants in the limit $\hbar \lambda^\phi \to 0$. In this limit, and with $\Delta \rho = 0$, one does recover eqs. (7) and (8). There are several interesting remarks concerning eqs. (18) and (19), but let us first point out that they do reduce to the known case (9), (10) in the limit, $n = 1$, where the multifractal moment is the resistance. Indeed, letting $n = 1$, $c_g(1) = r_g$, and $c_h(1) = r_h$ in (18) for example, we find, with $\phi = -x_1 - y_1$,

$$\langle R \rangle_p = r_g \lambda^{-x_1} G_p + r_h \lambda^{-y_1} (h \lambda^\phi)^2 \tilde{B}_p$$

$$= r_g \lambda^{-x_1} [G_p + (h \lambda^\phi) \tilde{B}_p] = r_g \lambda^{-x_1} \tilde{G}_p.$$  \hspace{2cm} (20)

In other words, $(h \lambda^\phi) \tilde{B}_p$ can be seen as a contribution from the expansion of $\tilde{G}_p$ in powers of $h \lambda^\phi$. The corresponding result for non-percolating systems follows in the same way.

Eqs. (18) and (19) for $\langle \delta R^n \rangle_p$ and $\langle \delta R^n \rangle_{NP}$ can be justified intuitively by noting that $c_g(n)$ and $c_h(n)$ are in principle independent microscopic noise characteristics which have no influence on the current distribution itself. So it is not surprising that they give independent contributions. The linear dependence on $c_g(n)$ and $c_h(n)$ of eqs. (18) and (19) is exact. The terms proportional to $(h \lambda^\phi)^{2n}$ can be understood by noting that the overall scale of current in the bad conductors of percolating samples will be down by $h \lambda^\phi$, which is the ratio between the resistance of good and bad conductors at scale $\lambda$. This ratio is given by the ratio between the resistance of percolating samples at scale $\lambda$ and non-percolating ones at the same scale.

That eqs. (18) and (19) for $\langle \delta R^n \rangle_p$ and $\langle \delta R^n \rangle_{NP}$ are the solution to the apparent contradiction of the previous section can be seen as follows. First, there is a single crossover exponent since $h$ always scale with $\lambda^\phi$. Second, the inescapable inequality of the results for percolating and non-percolating samples as $h L^\phi > 1$ is also verified. Indeed, setting $\Delta \rho = 0$, $\lambda = L$, and requiring that both eqs. (18) and (19) reproduce the Euclidean behavior $L^n = L^{2n - \beta (2n - 1)}$, we find that both equations take the form

$$\langle \delta R^n \rangle = L^n (A_g c_g h^{\nu - \phi} + A_h c_h h^{\nu + \phi}),$$  \hspace{2cm} (21)

where $A_g$ and $A_h$ are constants. The way out of the paradox of the previous section is thus that the existence of confluent singularities allows the behaviors above and below $\rho_c$ to match at finite $h$ with a single crossover exponent $\phi$. In other words, setting the noise from bad conductors $c_h(n)$ to zero in the scaling relations (18) and (19) and using analyticity, we find relations between the crossover functions $G_p$ and $\tilde{G}_{NP}$ but no new exponent relation. Similarly, when $c_g(n)$ vanishes, analyticity gives relations between $\tilde{B}_p$ and $\tilde{B}_{NP}$. 
The complete description of the multifractal moments for \( n \geq 2 \) thus involves for each value of \( n \) the basic exponents \( \nu \) and \( \phi \), as well as two new exponents \( -x_n - y_n \), and four crossover functions: \( G_p, B_p \) for the percolating samples (i.e. above \( p_c \) in the infinite size limit), and \( G_{NP}, B_{NP} \) for non-percolating samples (below \( p_c \) in the infinite size limit).

While one may be tempted to associate the exponent \(-x_n - y_n\) with the ratio \( c_e(n)/c_n(n) \), this parameter can vary from 0 to \(+\infty\) without influencing any fixed point or renormalization group flow. It is thus not a parameter in a renormalization group. Associating a crossover exponent to it is somewhat artificial. It always enter linearly in eqs. (18), (19). The ratio \( c_e(n)/c_n(n) \) was taken equal to \((r_e/r_n)^n\) in ref. [5]. This is also what one expects for the physical case of noise since \( c_e/r^n \) for example is equal to \( \langle \delta r_e^n \rangle / r^n \), the microscopic \( n \)th resistance fluctuation cumulant, normalized by the resistance. This quantity depends very weakly on resistance in realistic situations. One should note that for this choice of \( c_e(n)/c_n(n) = h^n \), the two exponents in eq. (21) become identical [5] in the self-dual two-dimensional case since then the equality \( x_n - nx_1 = -y_n + ny_1 \) is satisfied exactly [18, 21].

5. First and second renormalization group for multifractals

The behavior of the multifractal moments \( G_p, B_p, G_{NP}, B_{NP} \) can be summarized by writing down a generalized homogeneous function for the joint probability distribution \( P \), as in ref. [8]:

\[
P((g_p(n)), (b_p(n)), (g_{NP}(n)), (b_{NP}(n)); p - p_c, h, L) = \lambda^{\delta \Sigma (x_n - x_n)} P((g_p(n) \lambda^{x_n}), (b_p(n) \lambda^{-x_n}), (g_{NP}(n) \lambda^{x_n}),
\]

\[\]

\[
(b_{NP}(n) \lambda^{-x_n}); (p - p_c) \lambda^{1/r_e}, h \lambda^0, L / \lambda).
\]

Quantities such as \( g_p(n) \) stand for several multifractal moments labeled by \( n \), the sum on the right-hand side being over the corresponding values of \( n \). As in ref. [8], the function \( P \) should be universal, except for non-universal metric factors multiplying every quantity on which \( P \) depends. We have not written these metric factors explicitly for the sake of clarity. The ensemble on which the joint probability distribution \( P \) is defined is the ensemble of realizations of the random resistance network. The expectation value of \( g_p \) is \( G_p, \) etc. These expectation values are calculated from the joint probability distribution \( P \) in eq. (22) with the measure \( \Pi, dg_p(n) db_p(n) dg_{NP}(n) db_{NP}(n) \). At the percola-
tion threshold, \( p = p_c, \ h = 0 \), and for finite systems, eq. (22) clearly exhibits
the fact that expectation of \( g_p^2 \) for example scales as \( L^{-3 \alpha} \), or that the \( k \)th
cumulant for the fluctuations of \( g_p \) from one realization of the disorder to
another scales as \( L^{-k \alpha} \). This is the analog of gap scaling [8]. The analog of
universal amplitude ratios has also been discussed in ref. [8]. Note that only
positive integer multifractal moments are considered since these are the
macroscopically observable ones. They also provide in principle a complete
description of the problem [8] even though a given quantity may converge
slowly [25]. We have added a semi-colon before the last three variables in the
expression for the joint probability distribution \( \mathcal{P} \) in eq. (23) to identify them
as special, a point which we discuss below.

We now address the question of the correlation length again. What we claim
is that the correlation length is determined by the last three variables of \( \mathcal{P} \) in
eq (22). The other variables are described by a second renormalization group
which is slaved to the first one. It is the first one which determines the
correlation length for both renormalization groups. Although the ideas which
we now discuss have been suggested on rather general grounds [6, 9, 10], and
for 6-\( \varepsilon \) expansion at \( h = 0 \) [10], their explicit realization has also been seen in a
simple Migdal–Kadanoff (MK) renormalization group (RG) approach [18].
The first RG consists of the recursion relations for \( p \) and for \( h \). This set of
recursion relations for \( p \) and \( h \) is closed. If one goes beyond simple MK
renormalization, there should appear many more irrelevant operators, with
their corresponding fields. In the end, it is only the two fields, \( p \) and \( h \) in the
MK context, which determine the correlation length. The multifractal moments
for \( n \geq 2 \) belong to a second RG which is slaved to the first one. In other
words, none of the operators of the second RG influence the position of the
fixed point of the first RG or the value of the correlation length and crossover
exponents. On the other hand, quantities such as \( p_c \), which are determined by
the first RG, come in the determination of the exponents \( x_n \) and \( y_n \) (\( n \geq 2 \)) of
the second KG. As noted in ref. [8], positive integer values of \( n \) suffice in the
sense that integer moments determine the current (squared) distribution. The
operators associated with the exponents \( x_n \) and \( y_n \) (\( n \geq 2 \)) are called dominant.
The corresponding fields in the MK context would be \( G_p \) and \( B_{np} \). There are
also subdominant operators which are the analog of irrelevant operators. In the
MK context, the non-linear scaling field \( h(L)^{2a} B_{np} \) involves \( h(L) \), a field of
the first RG, but \( B_{np} \) does not feed back to the first RG. The names dominant and
subdominant are motivated by the necessity to indicate that operators are not
relevant in the sense of modifying the fixed point of the first RG to which they
are slaved, and by the fact that their value can be shifted from positive to
negative by a scale dependent change of current units which does not modify
observables [6, 9, 10].
6. Conclusion

We have seen, in the context of noise in percolating systems, that although the multifractal moments of the conductors in the conductor-insulator problem scale as $G_p(n, L, \Delta p = 0, h = 0) \sim L^{-3n}$, while those of the conductors in the superconductor-conductor problem scale as $B_{np}(n, L, \Delta p = 0, h = 0) \sim L^{-3n}$, it is possible to match the behavior at finite $h$ without introducing for $h$ an infinite set of crossover exponents. This is possible because there are two confluent singularities in $h$ for $n \geq 2$ which allow the behaviors above and below $p_c$ to match at finite $h$ with a single crossover exponent. This in turn has as a consequence that it is possible to define a single correlation length, in the same sense as in critical phenomena. Although crossover lengths defined from different multifractal correlation functions may take the form

$$\xi_n = \Delta p^{-\nu} F_n(h \Delta p^{-\nu_b}),$$

(23)

where the crossover function $F_n$ may depend on $n$, this is analogous to what happens in critical phenomena if we use either the order parameter or the energy correlation functions to define $\xi$ [23]. Along renormalization group eigendirections, the ratios of any two $\xi_n$ are constants, independent of the distance to the critical point. This should be contrasted to the case where to each $n$ would correspond a different exponent: in this case the lengths $\xi_n$ could not be kept in a constant ratio.

We have argued that the uniqueness of the correlation length is natural in the context where the multifractal moments for $n \geq 2$ are described by a second renormalization group which is slaved to the first one, which determines the current distribution and hence the correlation length [6, 9, 10]. Monte Carlo simulations for the case $\Delta p \neq 0$ would be interesting to confirm the general picture proposed for the joint probability distribution $\mathcal{P}$ in eq. (22). Some of these simulations are now being done [26].

Finally, we note that the uniqueness of the correlation length, as well as the second renormalization group approach, have been seen and discussed in detail in the context of dynamical systems as well [19].

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Appendix

There are perhaps many aspects of this work professor Fisher would object to. There is one however which we are convinced he does not like at all. Because of historical circumstances we will not get into, the exponents $x_n$ and $y_n$ are defined in such a way that they are all negative. This violates two rules: exponents should be positive and exponents should be denoted by Greek letters. To remedy this problem, and at the same time consider an experimentally more relevant limit, we let $L$ be large enough that we are in the Euclidean limit, and consider the ratio $\left\langle \delta R^n \right\rangle / R^n$ which does not depend on whether the multifractal moment is calculated at fixed current or fixed voltage. We obtain that

$$
\left\langle \frac{\delta R^n}{R^n} \right\rangle_z = L^{-x(n-1)} \left[ \frac{c_{\xi}}{r_{\xi}} (\Delta p)^{-\eta} g_{\xi}(h \Delta p^{-\nu}) + \frac{c_{\kappa}}{r_{\kappa}} (\Delta p)^{-\eta} b_{\kappa}(h \Delta p^{-\nu}) \right],
$$

where the subscripts $+$ and $-$ refer to $p - p_c > 0$ or $p - p_c < 0$, respectively, while

$$
\theta_n = \nu(d(n-1) - x_n + nx_1) \quad (n \geq 1),
$$

$$
\theta'_n = \nu(d(n-1) + y_n - ny_1) \quad (n \geq 1).
$$

The latter results can also be written as

$$
\theta_n' = \nu(d(n-1) - z_n + nz_1) \quad (n \geq 1) \quad (z_n = 2ny_1 - y_n).
$$

These exponents are those that would come in an experiment. Moreover, in their range of definition, $n \geq 1$, they satisfy the requirements of being Greek and positive. The latter, as well as the fact that $\theta_n$ and $\theta'_n$ are convex and increasing functions of $n$, can be shown from the corresponding properties [11, 12] for $-x_n$ and $-z_n$. In the special case $n = 2$ one recovers, as expected, the previously defined exponents [11, 12] $\kappa$ and $\kappa'$ for noise in the Euclidean
limit $(\theta_2 = \kappa, \theta'_2 = \kappa')$. Also, in the self-dual case there are degeneracies: $\theta_n = \theta'_n$. Evidently, $\theta_i = \theta'_i = 0$, and the latin sons (or ancestors) $t$ and $s$ give the missing exponents.

References


