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## Scaling behavior of multifractal-moment distributions near criticality

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**Résumé .** — On étudie ici par simulation Monte Carlo les fluctuations d'un échantillon à l'autre des moments multifractals de réseaux de résistances aléatoires près du seuil de percolation. Pour des systèmes de taille  $L$ , les fluctuations ne dépendent de  $\Delta p$  qu'à travers la variable d'échelle  $\Delta p L^{1/\nu}$ . A  $\Delta p = 0$ , ce n'est qu'à travers la variable d'échelle  $hL^\phi$  que ces fluctuations dépendent de  $h$ , le rapport entre les bonnes et les mauvaises résistances. Ces résultats sont en accord avec les prédictions d'une forme d'échelle précédemment proposée pour la distribution de probabilité conjointe des moments multifractals. Dans la direction  $\Delta p \neq 0$ , les fluctuations relatives sont maximales lorsque la longueur de corrélation du massif est de l'ordre de  $L$ .

**Abstract .** — Sample to sample fluctuations of the multifractal moments of percolating random-resistor networks are studied via Monte Carlo simulations. For systems of size  $L$ , these fluctuations depend on  $\Delta p$ , the deviation from the critical concentration, only through the scaled variable  $\Delta p L^{1/\nu}$ . At  $\Delta p = 0$ , these fluctuations depend on  $h$ , the ratio of the good and bad conductances, only through  $hL^\phi$ . This is consistent with a previously proposed scaling ansatz for the joint probability distribution of multifractal moments. In the  $\Delta p \neq 0$  direction, the relative fluctuations are largest when the bulk correlation length is of the order of  $L$ .

### 1. Introduction.

Much of the attention which multifractal objects [1-6] have attracted probably comes from the fact that previous experience with critical phenomena and the renormalization group did not lead one to believe that an infinite set of exponents might be needed to deal with physically important problems. The percolation transition, for example, was believed to be a standard second-order transition. It came as a surprise then that resistance fluctuations due to resistance

noise [5], or equivalently moments of the current distribution in percolation networks [6], were found to scale with exponents which depend on the order of the moment considered. Within the larger context of multifractal objects this has sometimes been interpreted as coming from the existence of an infinite number of correlation lengths [7, 8] hence questioning in turn the use of standard renormalization group approaches for these problems.

The picture which seems to emerge now instead, is that much of the standard phenomenology associated with scaling, critical phenomena and the renormalization group still applies. In particular, there is still one basic correlation length: in other words, any reasonable definitions of correlation lengths differ only by prefactors [9, 10]. The scaling of this correlation length is dictated by the "first renormalization group" which contains the usual relevant operators, such as, in the case of percolation,  $\Delta p = p - p_c$  and  $h$  the ratio of microscopic conductivities [10-12]. The infinite number of exponents comes about here because each of the moments, in the field-theory formulation [13-16], couples to a different symmetry breaking field [16-18]. The associated operators were called "dominant" because they span a space where a "second" renormalization group (RG) operates. This second RG is slaved to the first one and cannot modify the fixed point or exponent of the first RG [9, 16-18].

One of the satisfying aspects of this point of view is that it is to some extent generic. It was applied for example with success to dynamical systems, more specifically the circle map [19]. Because of similarities between the field-theory formulations [3, 13-16, 20] this approach should apply to percolation and to localization. Of course for the more difficult problems, such as turbulence or diffusion-limited aggregates, what should correspond to the first renormalization group is not even known, so we cannot tell if the preceding picture remains valid.

The phenomenology of the approach just discussed is summarized by the scaling properties of a joint probability distribution for the multifractal moments. Given the scaling properties of this universal joint probability distribution, one can find the scaling of all macroscopic observables, in analogy with the case of critical phenomena where the free energy plays this role. In this paper then, we first recall the proposed scaling of the joint probability distribution, and we then use Monte Carlo simulations to verify, in the context of percolation, some predictions of this scaling form which had not yet been put to the test.

## 2. Model and summary of known results.

Consider a two-component random resistance network where each bond has a resistance  $r_g$  (good conductors) with probability  $p$  and  $r_b$  (bad conductors) with probability  $1 - p$ . When  $r_g$  is finite and  $r_b$  infinite, we have the conductor-insulator problem, while when  $r_g$  is zero and  $r_b$  finite, we have the superconductor-conductor problem. Working in a finite system of size  $L$  near the percolation threshold  $p_c$ , the percolating samples (PS) and the non-percolating samples (NPS) are averaged separately. By a percolating sample, we mean a sample with a spanning cluster of bonds having resistance  $r_g$ , the smallest of the two.  $\langle \rangle_P$  and  $\langle \rangle_{NP}$  refer to averages over respectively the percolating and the non-percolating realizations of the random network. Multifractal moments are moments of the current distribution averaged over realizations of the random network, and they are called multifractal because of their peculiar scaling properties at the percolation threshold  $\Delta p \equiv |p - p_c| = 0$  and  $h \equiv r_g/r_b = 0$ , namely,

$$G_P(n, L, \Delta p = 0, h = 0) = \langle \sum_{\alpha(g)} (i_\alpha^2)^n \rangle_P \sim L^{-x_n} \quad (1)$$

$$B_{NP}(n, L, \Delta p = 0, h = 0) = \langle \sum_{\alpha(b)} (i_\alpha^2)^n \rangle_{NP} \sim L^{y_n} \quad (2)$$

At the most simple-minded level, we say we have multifractal behavior because we have the infinite sets of exponents  $-x_n$  and  $y_n$  that do not obey the familiar linear dependence on  $n$  encountered in the case of gap scaling. There are more subtle definitions [14] but we do not need to go into these details here. The quantities appearing above are called multifractal moments.  $G$ 's refer to multifractal moments of the good conductors and  $B$ 's to multifractal moment of bad conductors. More specifically, the multifractal moments of the currents (squared) in the bonds of type  $r_g$  and the multifractal moments of the currents (squared) in the bonds of type  $r_b$ , averaged over the PS, are noted  $G_P(n, L)$  and  $B_P(n, L)$  respectively; the corresponding quantities averaged over the NPS are noted  $G_{NP}(n, L)$  and  $B_{NP}(n, L)$ . These quantities are computed for a unit applied current  $I$ .

The behavior of the multifractal moments  $G_P, B_P, G_{NP}, B_{NP}$  can be summarized by writing down a generalized homogeneous function for the joint probability distribution  $\mathcal{P}$ , as in references [9-12]

$$\begin{aligned} &\mathcal{P}(\{g_P(n)\}, \{b_P(n)\}, \{g_{NP}(n)\} \{b_{NP}(n)\}; p - p_c, h, L) = \\ &\lambda^2 \sum^{(x_n - y_n)} \mathcal{P}(\{g_P(n)\lambda^{x_n}\}, \{b_P(n)\lambda^{-y_n}\}, \{g_{NP}(n)\lambda^{x_n}\}, \{b_{NP}(n)\lambda^{-y_n}\}; \\ &(p - p_c) \lambda^{1/\nu}, h\lambda^\phi, L/\lambda). \end{aligned} \tag{3}$$

Quantities such as  $\{g_P(n)\}$  stand for several multifractal moments labeled by  $n$ , the sum on the right-hand side being over the corresponding values of  $n$ . As in references [9-12], the function  $\mathcal{P}$  should be universal, except for non-universal metric factors multiplying every quantity on which  $\mathcal{P}$  depends. *We have not written these metric factors explicitly for the sake of clarity.* The ensemble on which the joint probability distribution  $\mathcal{P}$  is defined is the ensemble of realizations of the random resistor network. The expectation value of  $g_P$  is  $G_P$  etc... These expectation values are calculated from the joint probability distribution  $\mathcal{P}$  in (3) with the measure  $\prod_n dg_P(n) db_P(n) dg_{NP}(n) db_{NP}(n)$ . Note that only positive integer multifractal moments are considered since these are the macroscopically observable ones. When they are known exactly, they also provide in principle a complete description of the problem [11] even though a given quantity may converge slowly [21]. The semi-colon before the last three variables in the expression for the joint probability distribution  $\mathcal{P}$  is used to emphasize that these quantities are associated with the first renormalization group.

The expression for the joint probability distribution predicts, for example, that in the scaling region we should have,

$$G_P(n, L, \Delta p, h) = \lambda^{-x_n} G_P(n, L/\lambda, \Delta p \lambda^{1/\nu}, h\lambda^\phi). \tag{4}$$

This scaling form and the corresponding ones for  $G_{NP}, B_P$ , and  $B_{NP}$  at  $\Delta p = 0$  has been checked explicitly in references [12,22]. Another prediction of the joint probability distribution is that at the percolation threshold,  $p = p_c, h = 0$ , and for finite systems, the expectation of  $g_P(n)^2$  for example scales as  $L^{-2x_n}$ , while the  $k$ 'th cumulant for the fluctuations of  $g_P$  from one realization of the disorder to another scales as  $L^{-kx_n}$ . This is the analog of gap scaling [11]. This had been known for a long time [23] in the  $n = 1$  case where this property follows from the fact that the probability distribution for the resistance of a percolating sample depends only on the resistance divided by  $L^{-x_1}$ . The analog of universal amplitude ratios which follow from the scaling form of  $\mathcal{P}$  has been discussed in reference [11] where several universal ratios were obtained, at  $\Delta p = 0, h = 0$ , by Monte Carlo simulations on square and triangular lattices.

In the following, we test some predictions of the scaling form of the joint probability distribution at  $h = 0$  and  $\Delta p$  finite. More specifically, we compute

$$C(m, n; k, \ell) \equiv \frac{\langle g_P(m, L, \Delta p, h)^k g_P(n, L, \Delta p, h)^\ell \rangle_{\text{cum}}}{\langle g_P(m, L, \Delta p, h) \rangle^k \langle g_P(n, L, \Delta p, h) \rangle^\ell} \quad (5)$$

where the averages are cumulant averages, and where  $k$  and  $\ell$  are integers. At  $\Delta p = 0$ ,  $h = 0$  the scaling of the joint probability distribution predicts that the above ratios,  $C(m, n; k, \ell)$ , are size independent and universal. In other words, the  $C(m, n; k, \ell)$  are the universal amplitude ratios we just referred to [11]. At finite  $\Delta p$  and  $h = 0$ , the prediction is that  $C(m, n; k, \ell)$  are universal functions of the variable  $\Delta p L^{1/\nu}$  only, while at finite  $h$  and  $\Delta p = 0$  they are functions of  $hL^\phi$  only. It is these dependencies which are verified in the next section.

### 3. Numerical results.

Monte Carlo results are obtained for random (bond) resistance networks. The lattices are of size  $L \times L$  and a unit current is applied through two parallel bus bars of length  $L$ . To compute the current distribution, one inverts Kirchhoff's laws using a sparse matrix algorithm [24]. Only percolating samples are kept to compute statistics of the multifractal moment distribution. In all cases, we restrict ourselves to  $15 \leq L \leq 63$ . The lower bound for  $L$  in  $d = 2$  seems to be sufficiently large to be in the scaling regime, as discussed in reference [25]. We have also obtained results in  $d = 3$  which are similar to those presented below, but since the statistics are not as good we will not discuss this case any further.

Figure 1 shows the results for the average of the multifractal moments for  $n = 1, 2, 3$ . The various symbols stand for various values of  $\Delta p$ , as explained in the caption, and for a given  $\Delta p$  the multifractal moment increases with  $L$ . As expected from equation (4) with  $h = 0$  and  $\lambda = L$ , each moment, scaled by its size dependence in the fractal regime, is a function of the variable  $\Delta p L^{1/\nu}$  only. The flat portion of the plots represents the fractal regime. The slope on the right-hand side is  $\nu(x_n + d - 2n(d - 1))$ , as can be trivially obtained from the power law dependence  $d - 2n(d - 1)$  of the  $n$ 'th multifractal moment in the Euclidean regime. The slopes obtained numerically for  $n = 1, 2, 3$  respectively are,  $-1.3$ ,  $-3.8$ ,  $-7.0$ , while the corresponding predictions are,  $-1.30$ ,  $-3.76$ , and  $-6.36$ . The corresponding results for  $\Delta p = 0$  and  $h$  finite were presented in references [10, 12].

Figure 2 shows  $C(n, n; 1, 1)$  as defined in equation (5) for  $n = 1, 2, 3$  and  $h = 0$ . A large number of samples is needed to obtain accurate statistics for these quantities: from 20,000 for  $L = 15$ , to 8,000 for  $L = 63$ . Figure 2 is consistent with  $C(n, n; 1, 1)$  being also a function of the variable  $\Delta p L^{1/\nu}$  only, as predicted by the general scaling *ansatz* (3). At  $\Delta p = 0$ , it has been shown in a previous paper [11] that the results do not depend on whether the calculations are done for square or triangular lattice. Some of the calculations of reference [11] have been repeated for a check. The surprising result here, where  $\Delta p \neq 0$ , is the clear maximum in the  $n = 2$ , and  $n = 3$  cases. This maximum occurs when the bulk correlation length is of the order of the system size, namely  $\Delta p^{-\nu} \approx L$ . The corresponding results for  $\Delta p = 0$  and  $h$  finite are shown in figure 3. There is scaling also but no maximum this time.

To understand the occurrence of a maximum for the finite  $\Delta p$  case, we note that the special case studied here, namely  $C(n, n; 1, 1)$ , represents the variance of the distribution for the  $n$ 'th multifractal moment over the realizations of the disordered lattice. Histograms show that the relative width of the distribution is very large in the fractal regime  $\Delta p L^{1/\nu} \ll 1$ , while it is much smaller in the Euclidean regime  $\Delta p L^{1/\nu} \gg 1$ . The latter regime is self-averaging, so

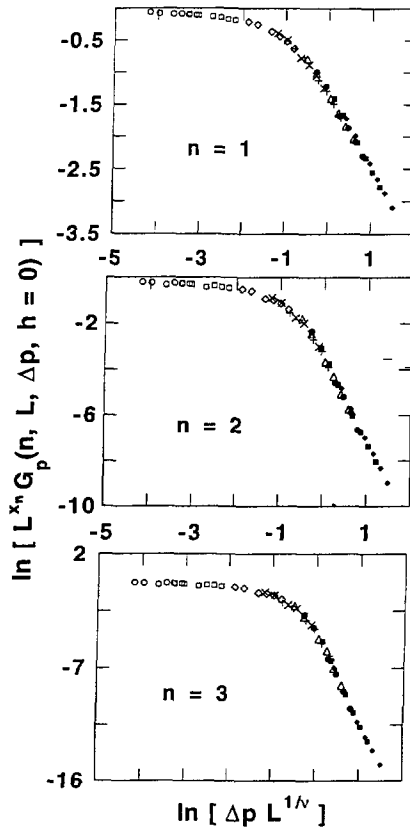


Fig. 1. — Log-Log plot of the first three multifractal moments, averaged over sample realizations and normalized to their behavior in the fractal regime, as a function of  $(\Delta p L^{1/\nu})$ . Each symbol is associated with a single value of  $\Delta p$  and several system sizes, namely  $L = 15, 20, 31, 40, 50, 63$  for (o)  $\Delta p = 0.002$ ; ( $\square$ ) 0.005; ( $\diamond$ ) 0.02; ( $\times$ ) 0.04; (+) 0.06; ( $\Delta$ ) 0.08; ( $\bullet$ ) 0.1; ( $\blacksquare$ ) 0.15; ( $\blacklozenge$ ) 0.2. The exponents used in the normalization were,  $x_1 = -0.98$ ,  $x_2 = -0.82$ ,  $x_3 = -0.77$ ,  $\nu = 4/3$ .

the relative width vanishes in the infinite system-size limit, as is clear from figure 2. In the intermediate regime,  $\Delta p L^{1/\nu} \approx 1$  we suspect that the few samples which have a fractal-type behavior influence the fluctuations more strongly than they influence the mean, leading to the observed maximum in the relative fluctuations. Indeed, one can check that after dividing out the fractal scaling behavior  $L^{-2x_2}$  from the numerical values of the numerator (fluctuations) entering the definition (5) of  $C(2, 2; 1, 1)$ , we do see a monotonous decrease similar to that in figure 1. However, as we leave the fractal regime, this decrease is not as fast as the decrease of the similarly normalized denominator (mean), leading to the observed maximum in the ratio. It is in this intermediate regime then that the relative fluctuations are largest. The maximum does not appear in the  $\Delta p \approx 0$ ,  $h$  finite case probably because the physical origin of the finite correlation length in this case is very different: this length can be varied without changing the geometry of the network.

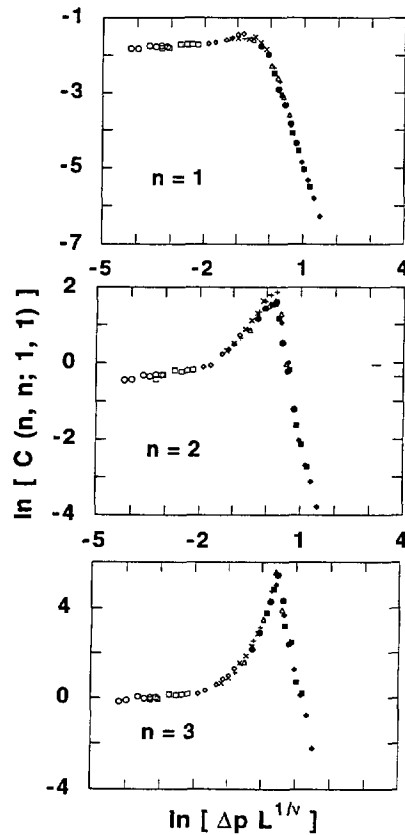


Fig. 2. — Logarithmic plot of some amplitude ratios, defined by equation (5), for the first three multifractal moments at  $h = 0$ . The symbols are as in figure 1.

#### 4. Conclusion.

Monte Carlo simulations have allowed us to complete the verification of the predictions of the general scaling *ansatz* (3) for the joint probability distribution of multifractal moments in finite-size random resistor networks near the percolation threshold. More specifically, we have checked that for the metal-insulator mixture problem (percolating samples), the ratio of the mean multifractal moments and their second cumulants depend on  $\Delta p$  only through  $\Delta p L^{1/\nu}$  when  $h = 0$ , while they depend on  $h$  only through  $h L^\phi$  when  $\Delta p = 0$ . Similar results are expected for all ratios in equation (5) and their generalizations. The superconductor-metal mixture should also give similar results. In the finite  $\Delta p$  case, the relative fluctuations are maximum in the intermediate regime where the bulk correlation length is of the order of the system size. We expect that analogous results will hold in the context of localization [3, 20] and dynamical systems [19].

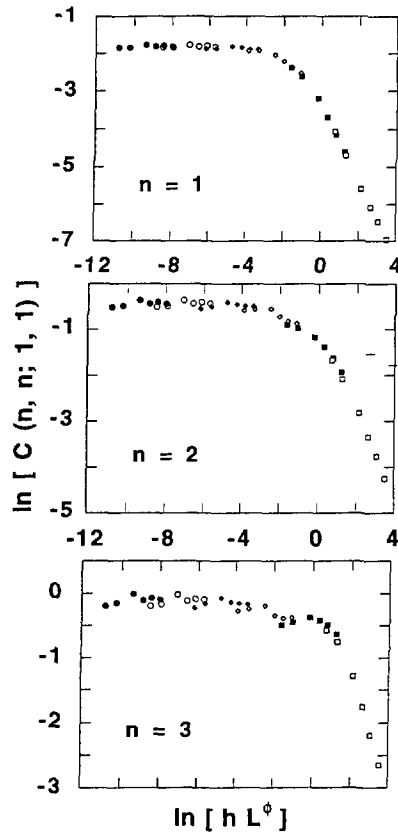


Fig. 3. — Logarithmic plot of the same amplitude ratios as in figure 2, this time at  $\Delta p = 0$ , as a function of  $hL^\phi$  with  $\phi = 1.96$ . Each symbol is associated with a single value of  $h$  and several system sizes, namely  $L = 15, 20, 31, 40, 50, 63$  for  $(\bullet)h = 10^{-7}$ ;  $(\circ)10^{-6}$ ;  $(\blacklozenge)10^{-5}$ ;  $(\diamond)10^{-4}$ ;  $(\blacksquare)10^{-3}$ ;  $(\square)10^{-2}$

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**References**

[1] MANDELBROT B.B., Proceedings of the Thirteen International Union of Pure and Applied Physics (IUPAP) Conference on Statistical Physics, D. Cabib, C.G. Kuper and I. Reiss Eds.

- (Hilger, Bristol, 1978).
- [2] HALSEY T.C., JENSEN M.H., KADANOFF L.P., PROCACCIA I. and SHRAIMAN B.I., *Phys. Rev. A* **33** (1986) 1141; **34** (1986) 1601;  
GRASSBERGER P., *Phys. Lett.* **97A** (1983) 227;  
HENTSHEL H.G.E. and PROCACCIA I., *Physica D* **8** (1983) 435.
- [3] IOFFE L.B., SAGDEEV I.R. and VINOKUR V.M., *J. Phys. C* **18** (1985) L641;  
CASTELLANI C. and PELITI L., *J. Phys. A* **19** (1986) L429.
- [4] HALSEY T.C., MEAKIN P. and PROCACCIA I., *Phys. Rev. Lett.* **56** (1986) 854.
- [5] RAMMAL R., TANNOUS C., BRETON P. and TREMBLAY A.-M.S., *Phys. Rev. Lett.* **54** (1985) 1718;  
RAMMAL R., TANNOUS C. and TREMBLAY A.-M.S., *Phys. Rev. A* **31** (1985) 2662;  
RAMMAL R. and TREMBLAY A.-M.S., *Phys. Rev. Lett.* **58** (1987) 415.
- [6] DE ARCANGELIS L., REDNER S. and CONIGLIO A., *Phys. Rev. B* **31** (1985) 4725;  
DE ARCANGELIS L., REDNER S. and CONIGLIO A., *Phys. Rev. B* **34** (1986) 4656.
- [7] FORGACS G., LIPOWSKY R. and NIEUWENHUIZEN T.M., to appear in "Phase Transitions and Critical Phenomena" C. Domb and J. Lebowitz Ed.;  
PALADIN G. and VULPIANI A., *Phys. Rev. B* **35** (1987) 2015.
- [8] DE ARCANGELIS L. and CONIGLIO A., *J. Stat. Phys.* **48** (1987) 935. Although the existence of an infinite number of correlation lengths is never mentioned explicitly in this paper, it would be implied by the infinity of crossover exponents discussed there. (See following reference).
- [9] TREMBLAY A.-M.S., TREMBLAY R.R., ALBINET G. and FOURCADE B., *Physica A* **183** (1992) 398.
- [10] TREMBLAY R.R., Ph. D. Thesis, Université de Sherbrooke, Québec, Canada (1992).
- [11] FOURCADE B., BRETON P. and TREMBLAY A.-M.S., *Phys. Rev. B* **36** (1987) 8935. For another approach to multifractal correlation functions, see also CATES M.E. and DEUTSCH J.M., *Phys. Rev. A* **35** (1987) 4907;  
SIEBESMA A.P., in *Universalities in Condensed Matter*, R. Jullien, L. Peliti, R. Rammal and N. Boccara Eds.;  
SIEBESMA A.P. and PIETRONERO L., *J. Phys.* **A21** (1988) 3259.
- [12] TREMBLAY R.R., ALBINET G., TREMBLAY A.-M.S., *Phys. Rev. B* **43** (1991) 11546 *Phys. Rev. B* **45** (1992) 755;  
MOROZOVSKII A.E., SNARSKII A.A., *JETP* **102** (1992) 683.
- [13] PARK Y., HARRIS A.B. and LUBENSKY T.C., *Phys. Rev. B* **35** (1987) 5048
- [14] DUPLANTIER B. and LUDWIG A.W.W., *Phys. Rev. Lett.* **66** (1991) 247.
- [15] EYINK G.L. (unpublished).
- [16] TREMBLAY A.-M.S. and FOURCADE B., XVII<sup>th</sup> International Colloquium on Group Theoretical Methods in Physics, Y. Saint-Aubin and L. Vinet Eds. (World Scientific, 1989).
- [17] FOURCADE B., Ph. D. Thesis, Université de Sherbrooke, Québec, Canada (1988).
- [18] FOURCADE B. and TREMBLAY A.-M.S., *Phys. Rev. B* **39** (1989) 6819.
- [19] FOURCADE B. and TREMBLAY A.-M.S., *J. Stat. Phys.* **61** (1990) 607; *ibid.* **61** (1990) 639;  
FOURCADE B. and TREMBLAY A.-M.S., *Phys. Rev. Lett.* **22** (1990) 2659.
- [20] WEGNER F., *Nucl. Phys. B* **280** [FS18] (1987) 210.
- [21] AHARONY A., BLUMENFELD R., BRETON P., FOURCADE B., HARRIS A.B., MEIR Y. and TREMBLAY A.-M.S., *Phys. Rev. B* **40** (1989) 7318.
- [22] KOLEK A., *Phys. Rev. B* **45** (1992) 205.
- [23] VANNIMENUS J. and NADAL J.P., *Phys. Rep.* **103** (1984) 47 and references therein.
- [24] Yale Sparse Matrix Package, EISENSTAT S.C., GURSKY M.C., SCHULTZ M.H. and SHERMAN A.H. (unpublished).
- [25] BOULAROT H. and ALBINET G., *Comptes-Rendus de l'Academie des Sciences de Paris* **314**, Série II (1992) 1119.