Universal topological phase of 2D stabilizer codes and decoding algorithms

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Outline

1. Kitaev’s code
   - Definition
   - Hamiltonian’s symmetries

2. Errors: defect creation, diffusion, and annihilation

3. Decoding problem
   - Task description
   - Perfect matching algorithm

4. Renormalization Group Decoder
   - Coarse graining
   - Mean-Field Equations

5. Results for Kitaev’s code

6. Local equivalence
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Lattice

- Two-dimensional square lattice
- Periodic boundary conditions
Site operator:
\[ A_s = \prod_{i \in V(s)} \sigma^x_i \]

Plaquette operator:
\[ B_p = \prod_{i \in V(p)} \sigma^z_i \]

Hamiltonian:
\[ H = - (\sum_s A_s + \sum_p B_p) \]
Kitaev's code

**Hamiltonian**

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- The Hamiltonian is a sum of commuting terms.
  - Exactly solvable
  - Constant gap
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Topological RG
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String operators

\[ Z_1 = \prod_{i \in \gamma_1} \sigma_z^i \]

- \([Z_1, B_p] = 0\]
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Second set of symmetries

- By reflecting around the diagonal, we obtain two new symmetry operators
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Non-trivial cycles
Non-trivial cycles
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Non-trivial cycles
Trivial cycles and ground space

- \( H = - (\sum_s A_s + \sum_p B_p) \)
- The \( A_s \) et \( B_p \) are trivial cycles
- Trivial action on ground space
  \[ A_s \ket{\psi} = B_p \ket{\psi} = +1 \ket{\psi} \]
- \( A_s \), \( B_p \) generate all trivial loops.

Trivial loops act trivially on ground space
Kitaev's code

Hamiltonian's symmetries

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Non-trivial cycles

- $\gamma_1$ and $\gamma_2$ wrap around the torus: they are non-trivial cycles
Kitaev’s code

Hamiltonian’s symmetries

Gauge choice

\[ |\psi\rangle = B_{p'} |\psi\rangle \]

\[ \bar{Z}_1 |\psi\rangle = \bar{Z}_1 B_{p'} |\psi\rangle \]

\[ Z_1 \equiv Z_1 B_{p'} \]

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\[ \equiv \bar{Z}_1 \prod_{\rho} B_{\rho} \]
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One degree of freedom associated to each non-trivial cycle.
- Operator in same homological class act identically on ground space.
- We encode the quantum information is those degrees of freedom:
  - The information can only be modified by topologically non-trivial operators.
  - Robust when \((\ell \to \infty)\)... ?
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5. Results for Kitaev’s code

6. Local equivalence
Consider error $E = \sigma^i_x$.

- $\sigma^i_x$ anti-commutes with adjacent plaquettes.
- $\sigma^i_x|\psi\rangle$ is a -1 eigenstate of $B_p$ and $B_{p'}$.
- Since $H = -(\sum_s A_s + \sum_p B_p)$, $\sigma^i_x$ costs 2 energy units.
- This error has created a pair of magnetic particles.
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Particle diffusion

New error occurs on neighboring qubit:

- Restores the sign of the middle plaquette
- Flips the sign of the right plaquette

No net energy cost: particle has moved
Errors: defect creation, diffusion, and annihilation

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No net energy cost: particle has moved
- Error chains are attached to particles, each with given energy.
- Particles can move around at no energy cost.
- Error chains can be stretched freely.
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An error can annihilate two particles.

The particle’s worldline is left behind after fusion.

Particle fusion can leave behind a worldline corresponding to a logical operation.

Memory corruption
Errors: defect creation, diffusion, and annihilation

Particle annihilation

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Memory corruption
The same story holds for $\sigma_z$ errors

These will create electrical particles located at the lattice’s vertices (plaquette of dual lattice).
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An error produces defects (error syndrome)

- Measure particle position, but not worldline.
- Many worldlines consistent with defects.
- Worldline with different homologies have different effect on ground space: MUST be distinguished.

**Decoding**

Infer worldline homology from particle location.

**15% Noise rate**
Decoding problem

Task description

Error syndrome & decoding

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Existing methods

Energy Minimization

- Find shortest path connecting all defects.
- Equivalent to minimizing energy of random bond Ising model.
- Edmonds’ perfect matching algorithm: $O(\ell^6)$

- It is very slow, $O(\ell^6)$, limited to lattices $\ell \approx 100$.
- It is not optimal:
  - Does not take into account the topological equivalence of errors.
  - Does not take into account correlations between magnetic and electric particles.

Depolarization error model

- Independent on every qubit.
- No error with probability $1 - p$.
- Error $X$, $Y$, or $Z$ with probability $p/3$. 
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Two possible pairings with different homologies
- First one has lower weight (Energy).
- Second one is highly degenerate (Entropy).

Optimal decoding
Homology class with lowest free energy \( F = E - TS \).
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Two possible pairings with different homologies

- First one has lower weight (Energy).
- Second one is highly degenerate (Entropy).

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Homology class with lowest free energy $F = E - TS$.

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6. Local equivalence
Scale invariance

- **Original $B_p$ checks**
  - Basis change (row operations on $C$)
  - Obtain scale invariant generators
  - Structure similar to a concatenated code.
  - Soft-decode each small block.
  - Pass information to next encoding level.
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**Concatenated code**

- Encoding one qubit in 3 qubits.

### Decoding

$P(L) = \sum'_{E} P(E)$ where

- Sum over $E$ equivalent to $L$ and with right syndrome.
- $P(E)$ given by error model.

- Encode each of these qubits...

### Decoding

- Compute error probability for each encoded qubit.
- Pass that probability to the next level up.
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Think of the Kitaev’s code as a concatenated code:
- It is made up of a bunch of small (open boundary) topological codes, joined into larger topological codes, etc.
- Given the particle configuration in a unit cell, compute the prob.
  - That it is transversed from top to bottom by a magnetic particle.
  - That it is transversed from top to bottom by an electric particle.
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  - etc.
- There are 16 possibilities corresponding to \(\{I, X, Y, Z\}^2\), 2 qubits.
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Topological codes are NOT concatenated codes.

- Unit cell does not enclose enough stabilizers.
- Area law increase of parameter space.
- Use overlapping cells instead.
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Self-consistence

- Boundary qubits treated as independent variables on neighboring unit cells.
- Probabilities assigned by different cells to a given qubit differ.
- Impose mean-field consistencies conditions on marginal probabilities.
- Solve by belief propagation.
- Complexity $O(\ell^2)$ parallelizable to constant time.
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6. Local equivalence
- Threshold $\approx 15\%$, compared to 15.5\% for PMA.
- Exponentially faster with marginal performance loss.
Results for Kitaev’s code

Smaller unit cell, $2 \times 1$

- Bit-flip threshold $\approx 8.2\%$, compared to $10.3\%$ for PMA.
- Much faster even without parallelization ($10^6$ sites).
- Illustrates flexibility.
Use of additional belief propagation.

Threshold $\approx 16.5\%$, compared to $15.5\%$ for PMA.

Illustrates flexibility.
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Conventional phases of matter are classified by order parameters and symmetry breaking (Landau-Ginsberg)

How to characterize and classify topological phases?
Chen, Gu, and Wen: local equivalence of ground-state manifold

\[ U P_0 U^\dagger = P'_0 \]

If \( |\sigma| \leq r \), then \( |U\sigma U^\dagger| \leq w + c \) (finite light cone).
\( P_0 \) and \( P'_0 \) are adiabatically connected.
Hard to determine (not constructive).

If two topological codes are in the same phase

Switch between them fault tolerantly during computation
Virtually switch between them for the sake of decoding
Error model remains local
Stabilizers remain local, reliable error syndrome
Topological phase

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Consider two syndromes patterns $s_1$ and $s_2$ supported on a finite region $R$. These syndromes associate a topological charge to $R$. The charges of $s_1$ and $s_2$ are identical if there exists a transformation on $R$ that takes $s_1$ to $s_2$. Topological charges are their own anti-particle. String operators between two identical charges.
Consider two syndromes patterns $s_1$ and $s_2$ supported on a finite region $R$.

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Qubits located on vertices

Plaquette stabilizers $S_p = \bigotimes_{j \in \partial p} \sigma_j$, for
$\sigma = \sigma^x$ and $\sigma^z$.

16 topological charges
- 10 bosons
- 6 fermions

Same particle type and statistics as 2 copies of Kitaev’s code
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Obtained by identifying topological charges

- Local stabilizers $\Rightarrow$ Local stabilizers
- Syndrome information directly available
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Results

Decoding error probability vs Bit-Flip channel strength $p\%$ for different values of $l$.
Every 2D, translationally invariant, non-chiral stabilizer code with local generators and macroscopic minimal distance is locally equivalent to a finite number of copies of Kitaev’s code.
**Topological subsystem code**

- **Qubits located on vertices**
- **Gauge generators on edges** $\sigma \otimes \sigma$
  - $\sigma = \sigma^x, \sigma^y, \sigma^z$ red, green blue
- **Stabilizers = center of gauge group**
  - Has local generators
- **Transversal Clifford group**
- **Only 3 semionic fermions**
- **Matches a subset of** $2 \times$ Kitaev’s code
  - $f_1 \leftrightarrow (f, 0)$
  - $f_2 \leftrightarrow (e, f)$
  - $f_3 \leftrightarrow (m, f)$
- **Locally equivalent to at least 2 copies of Kitaev’s code.**
- **Only a subset of topological charges carry information, other are gauge.**
- **Not all topological charges are stabilized: only those that topologically interact with $f_1$, $f_2$, and $f_3$.**
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Topological RG
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  - Only 3 semionic fermions
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Topological subsystem code

- Qubits located on vertices
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Results

Decoding error probability vs. Depolarizing channel strength, for different values of $l$. The graph shows a clear trend where the decoding error probability increases with the depolarizing channel strength for each $l$. The data points for $l=8$, $l=16$, $l=32$, $l=64$, and $l=128$ are clearly visible, indicating a consistent pattern across different values of $l$. The legend indicates the line styles and colors corresponding to each value of $l$. The graph helps in understanding the relationship between the decoding error probability and the depolarizing channel strength, which is crucial for optimizing error correction strategies in quantum computing.
Decoding problem: infer defect worldline homology from "snapshots" of their configuration.

- Equivalent to minimizing free energy of some spin model.
  - Minimizing energy can be done efficiently for Kitaev’s code (RBIM).
  - Suboptimal, slow.

- RG decoding algorithm
  - Exponentially faster.
  - Versatile (other codes, time/performance tradeoff).
  - Higher threshold.
  - Heuristic (Bravyi has proved a threshold... $10^{-22}$)

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