Tradeoffs Between Thermal and Quantum Fluctuations in 2D Quantum Memories

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Classical memories are robust

- Energy barrier $\propto \sqrt{n}$ between logical states through local moves.
- Boltzmann: configuration $x$ has probability $\propto \exp(-E(x)/T)$.
- Probability of flipping the whole configuration by local moves decreases with $n$. 
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System has two ground states $|\uparrow\uparrow \ldots \uparrow\rangle$ and $|\downarrow\downarrow \ldots \downarrow\rangle$. 

- $\alpha|\uparrow\uparrow \ldots \uparrow\rangle + \beta|\downarrow\downarrow \ldots \downarrow\rangle$ does not evolve in time.

- Local observable $\sigma_i^z$ distinguishes them.

- Local order parameter $\sigma^z$.

- Local perturbation $B\sigma_z$ lifts degeneracy:

$$\alpha|\uparrow\uparrow \ldots \uparrow\rangle + \beta|\downarrow\downarrow \ldots \downarrow\rangle \xrightarrow{t} e^{-iBt}\alpha|\uparrow\uparrow \ldots \uparrow\rangle + e^{iBt}\beta|\downarrow\downarrow \ldots \downarrow\rangle$$

Unknown $B$:

$$\begin{pmatrix}
|\alpha|^2 & e^{-i2Bt}\alpha^*\beta \\
e^{i2Bt}\alpha\beta^* & |\beta|^2
\end{pmatrix}
\xrightarrow{\int dB}
\begin{pmatrix}
|\alpha|^2 & 0 \\
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\end{pmatrix}$$

- Quantum superposition $\rightarrow$ Statistical mixture.
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The system has a stable spectrum.
Long lived memory at zero temperature.

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H = \sum_i \sigma_i^z \sigma_{i+1}^z + \sigma_{23}^z
\]

The ground state manifold changes abruptly when including site 23.

Can we combine this spectral stability with the thermal stability of the 2D Ising model?

In this talk: some evidence that it cannot be done in 2D.
Topological quantum order

Bravyi, Hastings, & Michalakis

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(TQO2) System is locally consistent.

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Outline

1. 2D Commuting Projector Codes
2. Holographic Disentangling Lemma
3. Holographic Minimum Distance
4. String-Like Logical Operators
5. Thermal instability
6. Open Questions
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1. 2D Commuting Projector Codes
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Definitions

- $\Lambda$ is a 2D lattice.
- Each vertex occupied by $d$-level quantum particle.
- Hamiltonian $H = -\sum_{X \subseteq \Lambda} P_X$ with
  - $P_X = 0$ if radius($X$) $\geq w$.
  - $[P_X, P_Y] = 0$.
  - $P_X$ are projectors (optional).
- Code $C = \{\psi : P_X |\psi\rangle = |\psi\rangle\}$
  - = ground space of $H$
  - = image of code projector $\Pi = \prod_X P_X$
- With proper coarse graining, we can assume that
  - $\Lambda$ is a regular square lattice.
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Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
- Levin & Wen’s string-net models
- Turaev-Viro models
- Kitaev’s quantum double models
- Most known models with topological quantum order

Remark
The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark
Subsystem codes do not belong to this family.
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Correctable region

A region $M \subset \Lambda$ is \textit{correctable} if there exists a recovery operation $\mathcal{R}$ such that $\mathcal{R}(\text{Tr}_M \rho) = \rho$ for all code states $\rho$.

$M$ correctable $\iff$ No order parameter on $M \iff \Pi O_M \Pi \propto \Pi$.

Minimum distance

The minimum distance $d$ is the size of the smallest non-correctable region.

Logical operator

Operator $L$ such that $L|\psi\rangle$ is a code state for any code state $|\psi\rangle$. 

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Standard definitions

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Holographic disentangling lemma (Bravyi, DP, Terhal)

Let $M \subset \Lambda$ be a correctable region and suppose that its boundary $\partial M$ is also correctable. Then, there exists a unitary operator $U_{\partial M}$ acting only on the boundary of $M$ such that, for any code state $|\psi\rangle$,

$$U_{\partial M} |\psi\rangle = |\phi_M\rangle \otimes |\psi'_M\rangle$$

for some fixed state $|\phi_M\rangle$ on $M$.

Remark

For a trivial code $\text{Tr} \Pi = 1$, every region is correctable, so we recover the area law $S(M) \leq |\partial M|$ for commuting Hamiltonians of Wolf, Verstraete, Hastings, and Cirac.
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Let $M \subset \Lambda$ be a correctable region and suppose that its boundary $\partial M$ is also correctable. Then, there exists a unitary operator $U_{\partial M}$ acting only on the boundary of $M$ such that, for any code state $|\psi\rangle$,

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for some fixed state $|\phi_M\rangle$ on $M$.

Remark

For a trivial code $\text{Tr} \Pi = 1$, every region is correctable, so we recover the area law $S(M) \leq |\partial M|$ for commuting Hamiltonians of Wolf, Verstraete, Hastings, and Cirac.
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB} P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.

\[ \mathcal{H}_B = \bigoplus_J \mathcal{H}_{B^J_L} \otimes \mathcal{H}_{B^J_R} \text{ and } \Pi = \bigoplus_J P_{AB^J_L} \otimes P_{B^J_R \overline{M}} \]

This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.
- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that
  \[ \Pi = V_B P_{AB^1} \otimes P_{B^2 \overline{M}} V_B^\dagger. \text{ (*)} \]
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\bar{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB}P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.
- $\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B'_L} \otimes \mathcal{H}_{B'_R}$ and $\Pi = \bigoplus_J P_{AB'_L} \otimes P_{B'_R\bar{M}}$
- This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.
- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2\bar{M}} V_B^\dagger$. ($\star$)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.

Write $\Pi = P_{AB}P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.

- $\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B^I} \otimes \mathcal{H}_{B^R}$ and $\Pi = \bigoplus_J P_{AB^I} \otimes P_{BR}M$

This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.

- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2}M V_B^\dagger$. (*)

[$\overline{M} = \Lambda \setminus M$]
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
  - Write $\Pi = P_{AB}P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.
  - $\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B_J^L} \otimes \mathcal{H}_{B_J^R}$ and $\Pi = \bigoplus_J P_{AB_J^L} \otimes P_{B_J^R M}$
  - This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.
  - We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2 \overline{M}} V_B^\dagger$. (⋆)
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- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB} P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.

- $\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$ and $\Pi = \bigoplus_J P_{AB_L} \otimes P_{B_{R \overline{M}}}$

- This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.

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- Write $\Pi = P_{AB}P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.
- $\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B_L}^J \otimes \mathcal{H}_{B_R}^J$ and $\Pi = \bigoplus_J P_{AB_L}^J \otimes P_{B_R}^J \overline{M}$
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- Write $\Pi = P_{AB} P_{B\overline{M}}$ with $[P_{AB}, P_{B\overline{M}}] = 0$.

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Holographic Disentangling Lemma

**Proof**

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{MC}P_{CD}$ with $[P_{MC}, P_{CD}] = 0$.
- $\mathcal{H}_C = \bigoplus_J \mathcal{H}_{C_L}^J \otimes \mathcal{H}_{C_R}^J$ and $\Pi = \bigoplus_J P_{MC_L}^J \otimes P_{C_R}^J D$
- This last sum over $J$ contains only one non-zero factor since $C \subset \partial M$ is correctable.
- We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC^1} \otimes P_{C^2 D} V_C^\dagger$. ($\star \star$)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{MC}P_{CD}$ with $[P_{MC}, P_{CD}] = 0$.

- $\mathcal{H}_C = \bigoplus_J \mathcal{H}_{C_J^L} \otimes \mathcal{H}_{C_J^R}$ and $\Pi = \bigoplus_J P_{MC_J^L} \otimes P_{C_R^J D}$
- This last sum over $J$ contains only one non-zero factor since $C \subset \partial M$ is correctable.
- We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC^1} \otimes P_{C^2 D} V_C^\dagger$. (⋆⋆)
Let $M$ be correctable.
Assume $\partial M$ is correctable.
Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
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$\mathcal{H}_C = \bigoplus_J \mathcal{H}_{C_L}^{J} \otimes \mathcal{H}_{C_R}^{J}$ and $\Pi = \bigoplus_J P_{MC_L}^{J} \otimes P_{C_R}^{J}D$

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We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC^1} \otimes P_{C^2D} V_C^\dagger$. (***)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{MC}P_{CD}$ with $[P_{MC}, P_{CD}] = 0$.

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This last sum over $J$ contains only one non-zero factor since $C \subset \partial M$ is correctable.

- We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC} C^1 \otimes P_{C^2 D} V_C^\dagger$. (**)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.

- Combining ($\ast$) with ($\ast\ast$), $\Pi' = V_B^\dagger V_C^\dagger \Pi V_B V_C = P_{AB}^1 P_{B^2C^1} P_{C^2D}$
- $P_{AB}^1 = |\eta_{AB}^1\rangle \langle \eta_{AB}^1|$ is rank one since $AB^1 \subset M$ is correctable.
- $P_{B^2C^1} = |\nu_{B^2C^1}\rangle \langle \nu_{B^2C^1}|$ is rank one since $B^2C^1 \subset \partial M$ is correctable.
- Let $V_{B^2C^1}$ be any unitary such that $V_{B^2C^1} |\nu_{B^2C^1}\rangle = |\alpha_{B^2}\rangle \otimes |\beta_{C^2}\rangle$.
- Then $U_{\partial M} = V_{B^2C^1} V_B^\dagger V_C^\dagger$ disentangles region $M$ as claimed.
Proof

- Let $M$ be correctable.
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Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
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Combining (⋆) with (⋆⋆), $\Pi' = V_B^\dagger V_C^\dagger \cap V_B V_C = P_{AB^1} P_{B^2 C^1} P_{C^2 D}$

- $P_{AB^1} = |\eta_{AB^1}\rangle\langle \eta_{AB^1}|$ is rank one since $AB^1 \subset M$ is correctable.
- $P_{B^2 C^1} = |\nu_{B^2 C^1}\rangle\langle \nu_{B^2 C^1}|$ is rank one since $B^2 C^1 \subset \partial M$ is correctable.
- Let $V_{B^2 C^1}$ be any unitary such that $V_{B^2 C^1}|\nu_{B^2 C^1}\rangle = |\alpha_{B^2}\rangle \otimes |\beta_{C^2}\rangle$.
- Then $U_{\partial M} = V_{B^2 C^1} V_B^\dagger V_C^\dagger$ disentangles region $M$ as claimed.
Proof

- Let $M$ be correctable.
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Outline

1. 2D Commuting Projector Codes
2. Holographic Disentangling Lemma
3. Holographic Minimum Distance
4. String-Like Logical Operators
5. Thermal instability
6. Open Questions
Holographic minimum distance (Bravyi, DP, Terhal)

Region $M \subset \Lambda$ is correctable if its boundary is smaller than the minimum distance $|\partial M| \leq cd$.

- Bulky errors are not problematic: it's the skinny ones we need to worry about.
- This hints at our next result: string-like logical operators.
Statement of the result

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Proof

Let \( M \subset \Lambda \) be a correctable region.

- If \(|\partial M| \leq d\), then \( \partial M \) is also correctable.
- Thus, we can reconstruct any code state \( \rho \) from \( \rho_{AD} = \text{Tr}_{\partial M} \rho \).
- But from the Holographic disentangling lemma, \( \rho_{AD} = \eta_A \otimes \rho_D \) with \( \eta_A \) independent of the encoded state \( \rho \).
- Thus, we can reconstruct \( \rho \) from \( \rho_D = \text{Tr}_{M \cup \partial M} \rho \), so \( M \cup \partial M \) is correctable.
- We can continue to grow \( M \) this way until \(|\partial M| \geq d\).

\[ \bar{M} = \Lambda \setminus M \]
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- Let $M \subset \Lambda$ be a correctable region.
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- But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with $\eta_A$ independent of the encoded state $\rho$.
- Thus, we can reconstruct $\rho$ from $\rho_D = \text{Tr}_{M \cup \partial M} \rho$, so $M \cup \partial M$ is correctable.
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String-like logical operators (Haah, Preskill)

There exists a non-trivial logical operator supported on a string-like region.

- Well known for Kitaev's toric code.
- Intuitive for known models that support anyons:
  - The ground state can be changed by dragging an anyon around a topologically non-trivial loop.
  - This process is realized on a string, and generated a logical operation.
- Relation to thermal instability?
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### Union of correctable regions

Let $M_1$ and $M_2$ be correctable distant regions and suppose that $\partial M_1$ is also correctable. Then, $M_1 \cup M_2$ is correctable.

- Trivial for syndrome-based error correction (e.g. stabilizer codes).
- We will prove the Knill-Laflamme condition $\Pi O_{M_1} \otimes O_{M_2} \Pi \propto \Pi$.
- The holographic disentangling lemma applied to $M_1$ implies that $\Pi = V_B V_C |\eta_{AB1}\rangle \langle \eta_{AB1}| \otimes |\nu_{B^2C^1}\rangle \langle \nu_{B^1C^1}| \otimes P_{C^2D} V^\dagger_B V^\dagger_C$.
- So $\Pi O_{M_1} \otimes O_{M_2} \Pi = f(O_{M_1}) \Pi O_{M_2} \Pi \propto \Pi$
- where $f(O_{M_1}) = \langle \eta_{AB1}| \langle \nu_{B^2C^1}| V^\dagger_B O_{M_1} V^B |\eta_{AB1}\rangle |\nu_{B^2C^1}\rangle$. 
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Let $\Pi_M = \prod_{X \cap M \neq \emptyset} P_X$

Then $X = \Pi_M O_M \Pi_M$ is a non-trivial logical operator supported on $M \cup \partial M$.

Any function of $X$, e.g. $\exp(-iX\theta)$, is also a logical operator with the same support.
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1. 2D Commuting Projector Codes
2. Holographic Disentangling Lemma
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4. String-Like Logical Operators
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**Noise model**

1. Apply random unitary on sites 1 & 2.
2. Measure $P_{12}$
   - If $P_{12} = 0$ go to 1.
3. Apply random unitary on site 3.
4. Measure $P_{23}$
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- Only a constant amount of energy at any given time.
- No need to backtrack.
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David Poulin (Sherbrooke)  2D Quantum Memories  Q+’13
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Analysis, part I

Define

\[ P_k = P_{k-1,k} \cdot P_{k-1,k} \]

\[ Q_k = (I - P_{k-1,k}) \cdot (I - P_{k-1,k}) \]

\[ D_k = \text{Depolarizing channel on } k \]

A typical step of the noise is \( P_k D_k Q_k D_k Q_k D_k Q_k D_k \).

Using the equalities

\[ D_k^2 = D_k \]

Average time

\[
\sum_{m=0}^{\infty} (m + 1) \text{Tr} \left[ P_k D_k (Q_k D_k)^m \rho \right]
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Average duration of an step depends only on the spectrum of a local operation \( B_k \), not on the lattice size.
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\mathcal{D}_k
\end{array}
\begin{array}{c}
Q_k \\
\mathcal{D}_k
\end{array}
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\[\exists \rho \text{ obtained at step } k \text{ supported on the subspace where } B_k = 1 \iff \text{Condition TQO2 is violated.}\]

Main result (Landon-Cardinal & DP)

The minimum set of conditions required to prove spectral stability imply the existence of a sequence of local maps that corrupt the system at an energy cost bounded by a constant.
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Open Questions

- **String-like logical operators +TQO ⇒ constant energy barrier.**
  - This is not directly related to thermal instability.
  - 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
  - What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
  - Can we characterize all string-like logical operators?
  - We have shown information corruption in time $\propto \sqrt{n}$. Can it be parallelized? (Percolation)
  - Relation between commuting projector codes and anyon models.

- Can we engineer dead ends?
  - Memory that is stabilized by complexity.

- Extension to subsystem codes?
  - With local stabilizer (Bombin) and without (Bacon-Shor).

- Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).
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String-like logical operators plus TQO imply a constant energy barrier.

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