A Relational Formulation of Quantum Theory

David Poulin

School of Physical Sciences, The University of Queensland
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  - GR says that physical descriptions should be background independent.

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  - QM sets a mathematical framework to describe physical systems: Hilbert space, unitary representations, etc.
  - GR says that physical descriptions should be background independent.

- We will also heavily rely on a Bayesian approach to quantum mechanics: Quantum states represent our knowledge about physical systems.

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To every “orthodox” physical description, we apply the following four rules:

1. Treat everything quantum mechanically.
2. Use Hamiltonians with appropriate symmetries.
3. Introduce equivalence classes between quantum states related by an element of the symmetry group.
4. Interpret diagonal entries of density operators as probability distributions.

In appropriate “macroscopic” limits, this description is equivalent to the orthodox description. Away from this limit, the relational description leads to new predictions. The orthodox description is an approximation to the fundamental relational description.
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- We will investigate the features of this theory.
  - New physical phenomenon.
  - Compare with more sophisticated relational theories: “experimental quantum gravity”.

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Outline

- Give orthodox description of a simple quantum mechanical system.
- Gradually apply our rules to arrive at a fully relational description.
  - Measurements.
  - Dynamics.
  - Time.
- Discussion
  - Relational time.
  - Fundamental decoherence.
  - Spin networks.
  - Connexion to other programs.
- Summary
Orthodox description

Spin-$\frac{1}{2}$ particle $S$ immersed in a magnetic field.

- Choose $\hat{x}$ such that $\vec{B} = B\hat{x}$.
- Hamiltonian $H^S = -B\sigma_x^S$.
- System’s initial state $|\psi(0)\rangle^S = \alpha|\uparrow\rangle^S + \beta|\downarrow\rangle^S$ in $\sigma_z$ basis.
- At time $t$,

$$
|\psi(t)\rangle^S = \alpha(t)|\uparrow\rangle^S + \beta(t)|\downarrow\rangle^S,
$$

$$
\alpha(t) = \alpha \cos(Bt/2) + i\beta \sin(Bt/2),
$$

$$
\beta(t) = i\alpha \sin(Bt/2) + \beta \cos(Bt/2).
$$
Orthodox description

To make a measurement at time $\tau$, we need a measurement apparatus $A$:
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- Initialize it in state $\left( |\uparrow_A \rangle + |\downarrow_A \rangle \right) / \sqrt{2}$. 

$H_{SA}(t) = g(t) S_z A$ with $g = 2$. 

At time $\tau$ immediately after, $S$ and $A$ are correlated:

$S = \left( |\uparrow \rangle + |\downarrow \rangle \right)_A = \left( |\uparrow \rangle \right)_S |\uparrow \rangle_A + \left( |\downarrow \rangle \right)_S |\downarrow \rangle_A$.

$S$ and $A$ are either both in up or both in down state, with respective probabilities $\left( |\uparrow \rangle \right)_S |\uparrow \rangle_A$ and $\left( |\downarrow \rangle \right)_S |\downarrow \rangle_A$. 

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Orthodox description

To make a measurement at time $\tau$, we need a measurement apparatus $A$:

- Initialize it in state $(|\uparrow\rangle^A + |\downarrow\rangle^A)/\sqrt{2}$.
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To make a measurement at time $\tau$, we need a measurement apparatus $\mathcal{A}$:

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- Coupling $H^{SA}(t) = -g\delta(t - \tau)\sigma_z^S \otimes \sigma_y^A$ with $g = \pi/2$.
- At time $\tau_+$ immediately after $\tau$, $S$ and $\mathcal{A}$ are correlated:

$$|\Psi(\tau_+)\rangle^{SA} = \alpha(\tau)|\uparrow\rangle^S \otimes |\uparrow\rangle^A + \beta(\tau)|\downarrow\rangle^S \otimes |\downarrow\rangle^A.$$
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To make a measurement at time $\tau$, we need a measurement apparatus $\mathcal{A}$:

- **Initialize it in state** $(|\uparrow\rangle^\mathcal{A} + |\downarrow\rangle^\mathcal{A})/\sqrt{2}$.
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|\Psi(\tau_+)\rangle^{S\mathcal{A}} = \alpha(\tau)|\uparrow\rangle^S \otimes |\uparrow\rangle^\mathcal{A} + \beta(\tau)|\downarrow\rangle^S \otimes |\downarrow\rangle^\mathcal{A}.
\]

- $\mathcal{A}$ is “classical”, so it collapses to either $|\uparrow\rangle^\mathcal{A}$ or $|\downarrow\rangle^\mathcal{A}$:

\[
\rho^{S\mathcal{A}} = |\alpha(\tau)|^2|\uparrow\rangle^S \otimes |\uparrow\rangle^\mathcal{A} + |\beta(\tau)|^2|\downarrow\rangle^S \otimes |\downarrow\rangle^\mathcal{A}.
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Orthodox description

This description is not background independent.

- Both $\vec{B}$ and $\mathcal{A}$ are treated classically, and refer to (or define) an external coordinate system.
  - Magnetic field $\vec{B} \propto \hat{x}$.
  - The observable $\sigma_z^A$ of $\mathcal{A}$ is superselected.
  - The coupling $H^{SA}(t)$ is neither time independent or rotationally invariant.

... so this is a pretty good model to illustrate our approach.
Some notation

Systems are labeled by a calligraphic capital letter, e.g. $\mathcal{A}$. 

$$J_{\mathcal{A}}^2 j_{\mathcal{A};i_{\mathcal{A}}} = \mathcal{A} (\mathcal{A} + 1) j_{\mathcal{A};i_{\mathcal{A}}} = a_{j_{\mathcal{A};i_{\mathcal{A}}}},$$

$$H_{\mathcal{A}} = C_{2\mathcal{A}+1}^2$$
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- Quantum number associated to total angular momentum is the same capital letter in roman font.
- Quantum number associated to the angular momentum along $z$ is the same lower case letter.
- For $\mathcal{A}$, a spin-$A$ particle, this gives
  \[
  (J^\mathcal{A})^2 |A, a\rangle^\mathcal{A} = A(A + 1) |A, a\rangle^\mathcal{A}
  \]
  \[
  J_z^\mathcal{A} |A, a\rangle^\mathcal{A} = a |A, a\rangle^\mathcal{A},
  \]
  \[
  |A, a\rangle^\mathcal{A} \in \mathcal{H}^{\mathcal{A}} = \mathbb{C}^{2A+1}.
  \]
To measure angular momentum, we need a gyroscope \( G \).
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Rule 1 says that it should be quantum mechanical, so to recover the orthodox result, we choose it to be in a coherent state $|G, g = G\rangle^G$. We abbreviate this $|G, G\rangle^G$.  

Define the TPCP map $E_{SG}: B(H_{SG}) \to B(H_{SG})$: 

$E_{SG}(\cdot) = Z_{SO(3)}R_{SG}\circ \rho_{SG}\circ R_{SG}\oplus\rho_{SG}\circ R_{SG}$;

$R_{SG}$ is the unitary representation of the rotation group on the pair $S \times G$.

d is the invariant Haar measure on $SO(3)$.

For all $\psi \in B(H_{SG})$, $E(\psi)$ is rotationally invariant.

Straightforward generalization to arbitrary number of systems.
Measurement

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- Define the TPCP map $\mathcal{E}^{SG} : \mathcal{B}(\mathcal{H}^{SG}) \rightarrow \mathcal{B}(\mathcal{H}^{SG})$:

$$\mathcal{E}^{SG}(\rho) = \int_{SO(3)} R^{SG}(\Omega) \rho R^{SG}(\Omega)^\dagger d\Omega,$$
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$$\mathcal{E}^{\mathcal{S}\mathcal{G}}(\rho) = \int_{SO(3)} R^{\mathcal{S}\mathcal{G}}(\Omega) \rho R^{\mathcal{S}\mathcal{G}}(\Omega)^\dagger \, d\Omega,$$

$R^{\mathcal{S}\mathcal{G}} = R^S \otimes R^G$ is the unitary representation of the rotation group on the pair $S - \mathcal{G}$.
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Straightforward generalization to arbitrary number of systems.
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- Denote $\rho_{R}$ a quantum state expressed in a reference frame $R$.
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- In an other reference frame $R'$, the same physical state is $\rho_{|R'} = R(\Omega)\rho_{|R}R(\Omega)^\dagger$ where $\Omega$ is the group element relating $R$ to $R'$. 
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- But what if we have no information about $R$, say because it doesn’t exists?
  - Without knowledge of $b$, the probability of $a$ is $p(a) = \sum_b p(a|b)p(b)$.
  - The group averaging procedure $\mathcal{E}$ is the exact analogue of this rule.
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- $R^{SG}$ is generated by the total angular momentum operator
  \[ \vec{J}^{SG} = \vec{\sigma}^S + \vec{J}^G. \]

- Above, $\vec{\sigma}^S = (\sigma_x^S, \sigma_y^S, \sigma_z^S)$ and $\vec{J}^G = (J_x^G, J_y^G, J_z^G)$ are the system and gyroscope angular momentum operators.
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- So it will be convenient to express the state of $S$ and $G$ in terms of $(J^{SG})^2$ and $J_z^{SG}$:

$$|\Psi\rangle^{SG} = (\alpha|\uparrow\rangle^S + \beta|\downarrow\rangle^S) \otimes |G, G\rangle^G$$

$$= \alpha|G+\frac{1}{2}, G+\frac{1}{2}; \frac{1}{2}; G\rangle + \frac{\beta}{\sqrt{2G+1}}|G+\frac{1}{2}, G-\frac{1}{2}; \frac{1}{2}; G\rangle + \frac{\beta\sqrt{2G}}{\sqrt{2G+1}}|G-\frac{1}{2}, G-\frac{1}{2}; \frac{1}{2}; G\rangle.$$
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- Quantum numbers: $(J^{SG})^2$, $J_z^{SG}$, $(\sigma^S)^2$, and $(J^G)^2$. 
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$$
\alpha|G+\frac{1}{2}, G+\frac{1}{2}; \frac{1}{2}; G\rangle + \beta \sqrt{2G+1} \left( \frac{\beta}{\sqrt{2G+1}} |G+\frac{1}{2}, G-\frac{1}{2}; \frac{1}{2}; G\rangle + \frac{\beta \sqrt{2G}}{\sqrt{2G+1}} |G-\frac{1}{2}, G-\frac{1}{2}; \frac{1}{2}; G\rangle \right).
$$

- Quantum numbers: $(J^{SG})^2$, $J_z^{SG}$, $(\sigma^S)^2$, and $(J^G)^2$.

- $(J^{SG})^2$, $(\sigma^S)^2$, and $(J^G)^2$ are rotationally invariant as they commute with the generator $\vec{J}^{SG}$. 
Measurement

Since $J_z^{SG}$ is the only operator depending on a coordinate system, the effect of $E^{SS}$ can be readily anticipated: it randomizes the associated quantum number and leaves the other ones unchanged

$$\left[ |\alpha|^2 + \frac{|\beta|^2}{2G + 1} \right] |G + \frac{1}{2}; \frac{1}{2}; G\rangle \langle G + \frac{1}{2}; \frac{1}{2}; G| \otimes \frac{112G+2}{2G + 2} + \frac{2G|\beta|^2}{2G + G} |G - \frac{1}{2}; \frac{1}{2}; G\rangle \langle G - \frac{1}{2}; \frac{1}{2}; G| \otimes \frac{112G}{2G},$$

Rule 4 gives the desired interpretation. When $G \neq 1$, we recover the orthodox result. This description is fully relational.
Measurement

Since $J^SG_z$ is the only operator depending on a coordinate system, the effect of $E^{SS}$ can be readily anticipated: it randomizes the associated quantum number and leaves the other ones unchanged:

$$\left[|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right]|G+\frac{1}{2}; \frac{1}{2}; G\rangle\langle G+\frac{1}{2}; \frac{1}{2}; G| \otimes \frac{112G+2}{2G+2} + \frac{2G|\beta|^2}{2G+G}|G-\frac{1}{2}; \frac{1}{2}; G\rangle\langle G-\frac{1}{2}; \frac{1}{2}; G| \otimes \frac{112G}{2G},$$

We can remove the quantum number associated to $J^SG_z$ from the physical description as it is always in a maximally mixed state, and hence carries no information.

$$\rho^{SG}_{\text{physical}} = \left[|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right]|G+\frac{1}{2}; \frac{1}{2}; G\rangle\langle G+\frac{1}{2}; \frac{1}{2}; G| + \frac{2G|\beta|^2}{2G+1}|G-\frac{1}{2}; \frac{1}{2}; G\rangle\langle G-\frac{1}{2}; \frac{1}{2}; G|.$$
Measurement

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$$\left|\alpha\right|^2 + \frac{\left|\beta\right|^2}{2G+1} \left|G + \frac{1}{2}; \frac{1}{2}; G\right\rangle\left\langle G + \frac{1}{2}; \frac{1}{2}; G\right| \otimes \frac{1_{2G+2}}{2G+2} + \frac{2G\left|\beta\right|^2}{2G+G} \left|G - \frac{1}{2}; \frac{1}{2}; G\right\rangle\left\langle G - \frac{1}{2}; \frac{1}{2}; G\right| \otimes \frac{1_{2G}}{2G},$$

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Rule 4 gives the desired interpretation.

When $G \rightarrow \infty$, we recover the orthodox result.

This description is fully relational.
Dynamics

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- We need a magnet $\mathcal{M}$ pointing in the $\hat{x}$ direction:

\[
|M, M\rangle_x^\mathcal{M} = \frac{1}{2^M} \sum_{m=-M}^{M} \left( \frac{2M}{M+m} \right)^{1/2} \langle M, m| \mathcal{M}.
\]
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\]

- Coupling must be a scalar function of $\vec{J}^\mathcal{M} \cdot \vec{\sigma}^S$.

- We choose the Heisenberg coupling

\[
H^{S,\mathcal{M}} = -2\lambda \vec{J}^\mathcal{M} \cdot \vec{\sigma}^S = -\lambda [ (J^{S,\mathcal{M}})^2 - (\sigma^S)^2 - (J^\mathcal{M})^2 ]
\]
The solution to Schrödinger’s equation of motion is

\[ |\Psi(t)\rangle^{SM} = |M, M\rangle^M_x \otimes |\psi(t)\rangle^S + C(t) \left[ \frac{1}{\sqrt{2M}} |M, M\rangle^M_x \otimes |\downarrow\rangle^S + |M, M - 1\rangle^M_x \otimes |\uparrow\rangle^S \right] \]

- \( |\psi(t)\rangle^S \) is the solution in the orthodox description with \( B = \lambda(2M + 1) \).
- \( C(t) = i\sqrt{M}2(\alpha - \beta)\sin(Bt/2)/(2M + 1) \) vanishes as \( M \to \infty \).
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- $|\psi(t)\rangle^S$ is the solution in the orthodox description with $B = \lambda(2M + 1)$.
- $C(t) = i\sqrt{M}2(\alpha - \beta) \sin(Bt/2)/(2M + 1)$ vanishes as $M \to \infty$.

This description still depends on an external coordinate system.
Dynamics

We reintroduce the gyroscope and apply the map $E^{SMG}$ to obtain

$$\rho_{\text{physical}}^{SMG} \approx \frac{1}{2M} \sum_{n=-M}^{M-1} \binom{2M}{M+n} |\Psi_n(t)\rangle\langle \Psi_n(t)|^{SMG}$$
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where

$$|\Psi_n(t)\rangle^{SMG} = \alpha(t)|G + \frac{1}{2} + n; G + \frac{1}{2}\rangle^{SMG} + \beta(t) \sqrt{\frac{M-n}{M+n+1}} |G + \frac{1}{2} + n; G - \frac{1}{2}\rangle^{SMG}$$

referring to the rotationally invariant quantum numbers $(J^{SMG})^2$ and $(J^{SG})^2$. 

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Dynamics

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$$\rho_{\text{physical}}^{SMG} \approx \frac{1}{2^{2M}} \sum_{n=-M}^{M-1} \binom{2M}{M+n} |\Psi_n(t)\rangle\langle\Psi_n(t)|^{SMG}$$

where

$$|\Psi_n(t)\rangle^{SMG} = \alpha(t)|\frac{1}{2} + n; G + \frac{1}{2}\rangle^{SMG} + \beta(t) \sqrt{\frac{M-n}{M+n+1}} |\frac{1}{2} + n; G - \frac{1}{2}\rangle^{SMG}$$

referring to the rotationally invariant quantum numbers $(J^{SMG})^2$ and $(J^{SG})^2$.

For clarity, the quantum numbers $(\sigma^S)^2 = S(S + 1)$, $(J^M)^2 = M(M + 1)$, and $(J^G)^2 = G(G + 1)$ have been omitted.
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- The binomian distribution is peaked around $n = 0$, with fluctuations of size $\Delta n \sim \sqrt{M}$.
- In this range, the term under the square root is $1 + O(1/\sqrt{M})$. 

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Thus, with probability approaching one as $M \to \infty$,

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for some random $n \in [-\sqrt{M}, \sqrt{M}]$. 


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The state of $S$ and $G$ obtained from tracing out the magnet is

$$\rho_{\text{physical}}^{SG} \approx |\alpha(t)|^2 |G + \frac{1}{2}; G; \frac{1}{2}\rangle\langle G + \frac{1}{2}; G; \frac{1}{2}| + |\beta(t)|^2 |G - \frac{1}{2}; G; \frac{1}{2}\rangle\langle G - \frac{1}{2}; G; \frac{1}{2}|.$$

Thus, with probability approaching one as $M \to \infty$,

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$$\rho^{SG}_{\text{physical}} \approx |\alpha(t)|^2|G + \frac{1}{2}; G; \frac{1}{2}\rangle\langle G + \frac{1}{2}; G; \frac{1}{2}| + |\beta(t)|^2|G - \frac{1}{2}; G; \frac{1}{2}\rangle\langle G - \frac{1}{2}; G; \frac{1}{2}|.$$

Can be established through direct calculations.
Dynamics

Thus, with probability approaching one as $M \to \infty$,

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Can be established through direct calculations.

Can use the fact that $[Tr_B, E^{AB}] = 0$ when the symmetry group acts unitarily on $B$, combined with the result concerning measurement.
Our description still uses an external time coordinate.
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- To keep tract of time, we need a clock $C$. 

Rule 1 says that it needs to be quantum mechanical $j_C; C i_C$.

We need a magnet $N$ to power this clock $j_N; N i_N$.

Heisenberg coupling $H_C N = 2 \sim J_C \sim J_N$.

$M = N$ should be an integer greater than 1, so that the clock's period is longer than the system's period.

We eliminate time using Rule 3 exactly as we did for orientation: 

$$T(t) = T_C Z_T C 0 U(t) U(t) y_d t.$$
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  - Heisenberg coupling $H^{CN} = -2\lambda \vec{J}_C \cdot \vec{J}_N$.
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- Denote \( \rho_{|T}(t) \) a quantum state at time \( t \), where the time refers to an external time coordinate system \( T \).
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- In an other coordinate system \( T' \), the same physical state is \( \rho_{|T'}(t) = e^{-iH\Delta} \rho_{|T} e^{iH\Delta} \) where \( \Delta \) is the time translation relating \( T \) to \( T' \).
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- In an other coordinate system $T'$, the same physical state is $\rho_{|T'}(t) = e^{-iH\Delta} \rho_{|T} e^{iH\Delta}$ where $\Delta$ is the time translation relating $T$ to $T'$.

- If we have no information about $T$, say because it doesn’t exists, Bayesian logic prescribes the group averaging procedure $T$. 

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We apply the same procedure as above:
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Quantum numbers $(J_{CG})^2$ and $(J_{SCG})^2$. 

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We apply the same procedure as above:

- Solve equations of motion for \( S, M, C, \) and \( N \).
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- Introduce gyroscope into the picture and perform rotation group average.
- Perform time average $\mathcal{T}$ and trace our magnets, or vice and versa.
- Remove maximally mixed non-relational degrees of freedom from physical description.

\[
\sum_c \sum_{s,s',r,r'} \frac{a_s a_s^*}{2} \frac{1}{2^{2C}} \sqrt{\left( \frac{2C}{C+c+\Lambda s} \right) \left( \frac{2C}{C+c+\Lambda s'} \right)} (-1)^{(r-1/2)(s-1/2)+(r'-1/2)(s'-1/2)} \\
\times \sum_u d^C_{u-r,c+\Lambda s} d^C_{u-r',c+\Lambda s'} |G + u; G + u - r\rangle \langle G + u; G + u - r'|^{SCG}
\]

- Quantum numbers $(J^{CG})^2$ and $(J^{SCG})^2$. 
To “read time”, we must measure $(J^C_G)^2$: this yields an outcome $G + u$.

Interpretation: Clock’s needle is at an angle $\theta = \cos^{-1}(u/C)$ with respect to $G$, so it’s $\theta$ o’clock.
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To "read time", we must measure \((JC^G)^2\): this yields an outcome \(G + u\).

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Conditioned on this outcome \(u\), we measure \((J^{SCG})^2\) and obtain the outcome \(G + u + s\) with probability \(P(s|\theta)\).

\(s = \pm 1/2\) is directly interpreted as the system’s spin relative to the gyroscope.
a) Probability distribution for the measurement outcome of $(J^{CG})^2$ for clock size $C = 20$. Dash line is $1/\pi \sqrt{C^2 - u^2}$ corresponding to a flat distribution of $\theta$. 
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b) Conditional probability of $(J^{SCG})^2$ indicating $s = -1/2$ for clock size $C = 20, 40, 100, \text{ and } 400$. Dash line indicate orthodox prediction.
What have we done so far?

- We have applied our four rules to the simple example. This forced us to...
  - Quantize the spacial and temporal reference frame, by introducing a quantum mechanical gyroscope and clock.
  - Quantize the external fields generating dynamics.
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- In the appropriate macroscopic limits, the predictions are the same as those of the orthodox theory.
- But this limit is an approximation to reality...
- We will now explore some features of this relational theory.
Relational time

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This differs from the usual Wheeler-DeWitt equation \(H|\Psi\rangle = 0\), which is a special case of our constraint.
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The relational time à la Page Wootters rises from classical correlations, not entanglement:

$$\sum_t |t\rangle\langle t|^{C} \otimes \rho^{S}(t)$$

instead of

$$\sum_t |t\rangle^{C} \otimes |\psi(t)\rangle^{S}$$
Fundamental decoherence

 Arrow of time?
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Bayesian approach: clock measurement used to estimate the coordinate time $p(t|\theta) = p(\theta|t)p(t)/p(\theta)$. 

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- Bayesian approach: clock measurement used to estimate the coordinate time $p(t|\theta) = p(\theta|t)p(t)/p(\theta)$.
- Given clock measurement outcome $\theta$, the state of $S$ is $\rho^S(\theta) = \int p(t|\theta)\rho(t)dt = \int p(t|\theta)e^{-iHt}\rho(0)e^{iHt}dt$.
Spin networks

To solve equations of motion, we first introduced two new operators $\vec{J}_1$ and $\vec{J}_2$ satisfying $\vec{J}^S + \vec{J}^M + \vec{J}_1 = 0$ and $\vec{J}^C + \vec{J}^N - \vec{J}_2 = 0$:

\[
\begin{align*}
\vec{J}^S & \quad \vec{J}_1 \\
\vec{J}^M & \quad \vec{J}^C + \vec{J}^N - \vec{J}_2 = 0
\end{align*}
\]
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To read the time, we introduced another operator $\vec{J}_3$ satisfying $\vec{J}^G + \vec{J}_2 + \vec{J}_3 = 0$, or in other words $\vec{J}_3 = -(\vec{J}^C + \vec{J}^N + \vec{J}^G)$. 
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To measure the system’s state relative to the gyroscope and clock, we introduced \( \vec{J}_{\text{total}} \) satisfying \( \vec{J}_1 + \vec{J}_3 + \vec{J}_{\text{total}} = 0 \), or equivalently \( \vec{J}_{\text{total}} = \vec{J}^S + \vec{J}^M + \vec{J}^C + \vec{J}^N + \vec{J}^G \).
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\begin{align*}
\vec{J}^S & \quad \vec{J}_1 & \quad \vec{J}^g & \quad -\vec{J}^C \\
\vec{J}^M & \quad \vec{J}_3 & \quad \vec{J}_2 & \quad -\vec{J}^N \\
\vec{J}_{total} &
\end{align*}
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The next step was the group average. On the diagram, this essentially boils down to removing the arrows!!!

- \(\vec{J} \to j\) such that \((\vec{J})^2 = j(j + 1)\).
- Not all edges are in an eigenstate of the operator \((\vec{J})^2\), so we need superposition of graphs.
- The amplitudes are linear functions of the non-relational amplitudes and Clebsch-Gordan coefficients.
Spin networks

This yields the diagram

\[
\sum_{j_{total}} P(j_{total}) \left[ \sum_{j_1 j_2 j_3} \alpha_{j_1 j_2 j_3} \right]
\]

Where \( P(j_{total}) \) stands for \( P_{j_{total}} \).

The final step was to perform a time average which imposed an energy superselection rule. This implies superselection of \( j_1 (j_1 + 1) + j_2 (j_2 + 1) \), which can be imposed by a Kronecker delta in the previous sum.
Spin networks

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Where \( \sum_i p_i [\Gamma_i] \) stands for \( \sum_i p_i |\Gamma_i\rangle \langle \Gamma_i| \).
Spin networks

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Each decorated graph $\Gamma$ is a **spin network** corresponding to a basis state of the relational theory.
Spin networks

- Each decorated graph $\Gamma$ is a spin network corresponding to a basis state of the relational theory.

- Vertex with edges $j_1$, $j_2$, and $j_3$ carries an intertwining operator $\mathbb{C}j_1(j_1+1) \otimes \mathbb{C}j_2(j_2+1) \otimes \mathbb{C}j_3(j_3+1) \rightarrow \mathbb{C}$, which in our simple model, give the Clebsch-Gordan coefficients required to remove the arrows.
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- The “sum over histories” $\mathcal{T}$, usually performed using spin foams, is here implemented at the level of $B(\mathcal{H})$ rather than $\mathcal{H}$.
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Connexions to other programs

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Define the interaction algebra $\mathcal{A} = Alg(\mathcal{G}) \simeq \bigoplus_J \mathcal{M}_{m_J} \otimes 1_{n_J}$.

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The relational degrees of freedom live in the $\mathcal{H}_J$.

These are also called the noiseless subsystems of the interaction algebra $\mathcal{A}$.

Spin networks form basis states for noiseless subsystems of collective noise channels (not necessarily $SU(2)$.)
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