I.2. Mean field theory of the paramagnetic to ferromagnetic transition

We consider a model Hamiltonian of interacting localized spins on a lattice in d dimensions. We restricted the orientation of spins along the z direction and the range of interaction to the nearest neighbors sites only. This is the Ising model whose Hamiltonian reads:

\[ H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \]

while for a ferromagnetic transition \( J_{ij} > 0 \). \( S_i = S_i^z \) is the spin operator along the z direction. Its eigenvalues are well known in the basis \( |\uparrow_i \rangle \):

\[ S_i |\uparrow_i \rangle = \pm \frac{1}{2} |\uparrow_i \rangle \]

so that one can simply identify \( S_i \) in the Hamiltonian as taking the values \( \pm \frac{1}{2} \). Now it is useful to normalize the spins to \( \pm 1 \), that is, one will consider

\[ H = -\sum_{\langle ij \rangle} J'_{ij} S_i S_j \]

where \( J'_{ij} = J_{ij} / 4 \). In the presence of a magnetic field the Hamiltonian reads

\[ H = -\sum_{\langle ij \rangle} J'_{ij} S_i S_j - \sum_i g \mu_B S_i \]

where we can define \( B' = g \mu_B \frac{1}{2} B = \mu_B B \) \( (g=2) \).

The calculation of the partition function in the canonical ensemble

\[ Z = \text{tr} e^{-\beta H} \]

can only be achieved in 1D \((d=1)\) and 2D \((d=2)\). In 3D one has to resort to approximants. The mean field theory is the simplest approach to 2 and to the calculation of critical
we will use the so-called *Variational Method* to derive the mean-field equations.

First recall the definition of statistical entropy for system described by the Hamiltonian $H$:

$$ S(D) = -k_B \text{Tr} \left\{ D \ln D \right\} $$

where $D = e^{-\beta H}$ is the density operator for the canonical ensemble. Also, the trace $\text{Tr}$ is defined

$$ \text{Tr} = \sum_1^N \langle s_i \ldots s_n | s_i \ldots s_n \rangle $$

where the $N$-spin state $| s_i \ldots s_n \rangle = | s_1 \rangle \otimes \cdots \otimes | s_N \rangle$ is a tensorial product of individual spin states.

Consider the identity

$$ S(D) < -k_B \text{Tr} D \ln D' $$

where $D \neq D'$

Now if the exact density operator cannot be calculated let us consider a trial, but simpler, Hamiltonian for which $S$ can be calculated exactly:

$$ H_e = -\sum_i x_i s_i $$

which is a one-body Hamiltonian, where $x_i$ is a variational parameter. The corresponding density operator is

$$ D_e = e^{-\beta H_e} $$

where $Z_e = \text{Tr} e^{-\beta H_e}$ is the trial partition function. We thus have the following inequality

$$ S_e(D) \leq -k_B \text{Tr} D_e \ln D_e $$
Here the equality is obviously satisfied for $D_E = D$. Multiplying the expression by $-T$ one finds:

\[
-TS(D_E) \geq \frac{K_BT}{ \text{Tr} D_E \ln \left( e^{-\beta H} \right) } \\
\geq \frac{K_BT}{ \text{Tr} \left\{ D_E (-\beta H) - D_E \ln 2 \right\} } \\
\geq -\frac{K_BT}{ \text{Tr} D_E \ln 2 } - \frac{K_BT}{ \text{Tr} D_E } \\
\]

Since the density is normalized by the partition function

\[
\text{Tr} D_E = 1
\]

So that

\[
-TS(D_E) \geq -\langle H \rangle_E - K_BT \ln 2 \\
\langle H_E \rangle - \langle H \rangle_E - TS(D_E) = F_E - \langle H \rangle_E \geq -\frac{K_BT \ln 2 - \langle H \rangle_E}{F_E} \\
\Rightarrow F_E \geq F + \langle H - H \rangle_E
\]

or

\[
F \leq F_E + \langle H - H \rangle_E
\]

which is known as the Bozollibor identity. We define

\[
F_{CM} [\beta, \{x_i\}] = F_E [\beta, \{x_i\}] + \langle H - H \rangle_E
\]

as the meanfield free energy that depends on $\beta$ and $x_i$ as natural variables. The variational method consists in the minimization of $F_{CM}$ with respect to $x_i$ in order to get the configurations $\{x_i^*\}$ that will lead to the best $F_{CM}$, that is closest to the exact $F$. Explicitly

\[
\left. \frac{\delta F_{CM}}{\delta x_i} \right|_{x_i = x^*} = 0
\]
The contributions to $F_{CM}$ are:

$$Z_E = \frac{\text{Tr} \ e^+ \beta \sum_i x_i \cdot s_i}{\prod_i \text{Tr} \ e^{\beta x_i s_i}}$$

$$= \prod_i \left[ \sum_{s_i=\pm 1} e^{\beta x_i s_i} \right]$$

$$Z_E = \prod_i \left[ e^{\beta x_i} + e^{-\beta x_i} \right] = 2^N \cosh^N (\beta \xi_i)$$

$$\langle H - HE \rangle_E = \frac{1}{Z_E} \text{Tr} \ (H - HE) \ e^{-\beta HE}$$

$$= -\sum_{\langle ij \rangle} J_{ij} \langle S_i \cdot S_j \rangle_E - \sum_i B_i \langle S_i \rangle_E + \sum_i x_i \langle S_i \rangle_E$$

As $HE$ is a one-body Hamiltonian,

$$\langle S_i \cdot S_j \rangle_E = \langle S_i \rangle_E \langle S_j \rangle_E$$

which leads to

$$\langle H - HE \rangle_E = -\sum_{\langle ij \rangle} J_{ij} \langle S_i \rangle_E \langle S_j \rangle_E - \sum_i (B_i - x_i) \langle S_i \rangle_E$$

Now

$$\langle S_i \rangle_E = \frac{1}{Z_E} \text{Tr} \ S_i \ e^{\beta x_i s_i}$$

$$= \frac{1}{Z_E} \left( e^{\beta x_i} - e^{-\beta x_i} \right)$$

$$\langle S_i \rangle_E = \frac{e^{\beta x_i} - e^{-\beta x_i}}{e^{\beta x_i} + e^{-\beta x_i}} = \tanh (\beta x_i)$$

It follows then

$$F_{CM} [\beta, \{x_i\}] = \sum_{\langle ij \rangle} J_{ij} \tanh (\beta x_i) \tanh (\beta x_j) - \sum_i (B_i - x_i) \tanh (\beta x_i) - \beta \xi_i \ln (2 \cosh \beta \xi_i)$$

Variational condition

$$\frac{\delta F_{CM}}{\delta x_i} |_{x_i = 0} = 0$$
\[
\frac{\delta F_{CM}}{\delta x_i} = -\sum_j J_{ij} \theta \beta x_j \frac{\partial \theta \beta x_i}{\partial x_i} - B_i \theta \partial \theta \beta x_i + x_i \theta \partial \beta x_i
\]

\[
= -\beta \theta \frac{\theta \beta x_i}{\partial x_i}
\]

\[
0 = -2\sum_j J_{ij} \theta \beta x_j \frac{\partial \theta \beta x_i}{\partial x_i} - B_i \theta \partial \theta \beta x_i + x_i \theta \partial \beta x_i
\]

\[
= \left[ -2\sum_j J_{ij} \theta \beta x_j^* - B_i \theta \beta x_i^* \right] \frac{\partial \theta \beta x_i}{\partial x_i} = 0
\]

\[
\Rightarrow \quad x_i^* = B_i + 2\sum_j J_{ij} \theta \beta x_j^*
\]

which can be considered as a self-consistent condition for the \(x_i^*\)s.

In the presence of a field \(B\), the natural variables of \(F_{CM}\) are \(\theta, B\) and \(x_i^*\). Both \(x_i^*\) and \(B\) are related through the self-consistent condition. In order to express the free energy in terms of the order parameter \(M\), let's do a Legendre transform

\[
F_{CM}[\theta, x_i^*, B] + \sum_i B_i M_i = F_{CM}[\beta, M_i]
\]

where the magnetization \(M_i\) is the conjugate variable of \(B_i\):

\[
M_i = -\frac{\partial F_{CM}}{\partial B_i} \quad \text{or} \quad \bar{M}_i = -\frac{\partial F_{CM}}{\partial B_i}
\]

because

\[
\bar{M}_i = -\frac{\partial F_{CM}}{\partial B_i} - \frac{\partial F_{CM}}{\partial x_i^*} \frac{\partial x_i^*}{\partial B_i}
\]

\[
\bar{M}_i = -\frac{\partial F_{CM}}{\partial B_i} = +\theta \beta x_i^*
\]

and

\[
\beta x_i^* = \ln \frac{\theta \beta}{1 - \bar{M}_i} = \frac{1}{2} \ln \left[ \frac{1 + \bar{M}_i}{1 - \bar{M}_i} \right] ; -1 < \bar{M}_i < 1
\]

\(F_{CM}[\beta, M_i] \) is called the Gibbs free energy, which will read...
$$\Gamma_{CM} [\beta, \bar{M}] = -\sum_{i<j} \mathbf{J}_{ij} \cdot \mathbf{\bar{M}}_{i} \cdot \mathbf{\bar{M}}_{j} - \sum_{c} B_{c} \bar{M}_{i} + \sum_{c} (B_{c} \bar{M}_{c})$$
\[+ \frac{1}{2} \sum_{c} \mathbf{\bar{M}}_{i} \cdot \ln \left[ \frac{1 + \mathbf{\bar{M}}_{i}}{1 - \mathbf{\bar{M}}_{i}} \right] - \beta^{-1} \frac{1}{2} \sum_{i} \ln (2 \cosh \beta x^{*}) \]

\[\text{if} \quad \ln 2 \cosh \beta x^{*} = \ln 2 + \frac{1}{2} \ln \cosh^{2} \beta x^{*} \]
\[\quad - \frac{1}{2} \ln (1 + \mathbf{\bar{M}}^{2} \beta x^{*}) \]
\[\quad - \frac{1}{2} \ln (1 - \mathbf{\bar{M}}^{2}) \]
\[\quad - \frac{1}{2} \left[ \ln (1 - \mathbf{\bar{M}}_{c}) + \ln (1 + \mathbf{\bar{M}}_{c}) \right] \]

$$\mathcal{M}_{CM} [\beta, \mathbf{\bar{M}}] = \frac{\mathcal{M}_{0}(\beta)}{\sqrt{\mathcal{K} \beta \Gamma}} - \sum_{i<j} \mathbf{J}_{ij} \cdot \mathbf{\bar{M}}_{i} \cdot \mathbf{\bar{M}}_{j} + \frac{1}{\beta} \sum_{c} \left[ \frac{(1 + \mathbf{\bar{M}}_{c}) \ln (1 + \mathbf{\bar{M}}_{c})}{2} \right]$$
\[+ \frac{(1 - \mathbf{\bar{M}}_{c}) \ln (1 - \mathbf{\bar{M}}_{c})}{2} \]

the first term $\sum \mathbf{\bar{M}}_{i} \mathbf{\bar{M}}_{j}$ is the internal energy, whereas the last two terms stand as entropy term.

If we consider the situation close to $T_c$ where $\mathbf{\bar{M}}_{c}$ is small ($\mathbf{\bar{M}}_{c} \ll 1$), one gets

$$\ln \left[ 1 + \mathbf{\bar{M}}_{c} \right] \approx \mathbf{\bar{M}}_{c} - \frac{\mathbf{\bar{M}}_{c}^{2}}{2} + \frac{\mathbf{\bar{M}}_{c}^{3}}{3} - \ldots$$

one gets for the entropy term:

$$\left( \frac{1 + \mathbf{\bar{M}}_{c}}{2} \right) \ln \left( 1 + \mathbf{\bar{M}}_{c} \right) + \left( \frac{1 - \mathbf{\bar{M}}_{c}}{2} \right) \ln \left( 1 - \mathbf{\bar{M}}_{c} \right)$$

$$\approx \left( \frac{1 + \mathbf{\bar{M}}_{c}}{2} \right) \left( \mathbf{\bar{M}}_{c} - \frac{\mathbf{\bar{M}}_{c}^{2}}{2} + \frac{\mathbf{\bar{M}}_{c}^{3}}{3} - \frac{\mathbf{\bar{M}}_{c}^{4}}{4} + \ldots \right)$$

$$+ \left( \frac{1 - \mathbf{\bar{M}}_{c}}{2} \right) \left( -\mathbf{\bar{M}}_{c} + \frac{\mathbf{\bar{M}}_{c}^{2}}{2} - \frac{\mathbf{\bar{M}}_{c}^{3}}{3} + \frac{\mathbf{\bar{M}}_{c}^{4}}{4} - \ldots \right)$$

$$= \mathbf{\bar{M}}_{c}^{2} (1 - \frac{1}{2}) + \mathbf{\bar{M}}_{c}^{4} \left( \frac{1}{6} - \frac{1}{8} - \frac{1}{8} \right) + \ldots = \mathbf{\bar{M}}_{c}^{2} + \frac{1}{12} \mathbf{\bar{M}}_{c}^{4} + \ldots$$
Up to quartic order one finds:
\[ \Gamma_{CM} [\beta, \bar{M}] = \Gamma_0 - \frac{1}{\langle i,j \rangle} \sum_j J_{ij}' \bar{M}_i \bar{M}_j + \beta^{-1} \sum_i \left( \frac{\bar{M}_i^2}{2} + \frac{1}{12} \bar{M}_i^4 + \ldots \right) \]

The uniform mean field solution is obtained by neglecting spatial variation of the order parameter \( \bar{M}_i \) by taking \( \bar{M}_i = \bar{M} \) for all sites \( i \). On these grounds:
\[ \bar{M} \frac{\Gamma_{CM}}{N} [\beta, \bar{M}] = \frac{1}{N} \sum_{i,j} J_{ij}' \bar{M}_i \bar{M}_j = \Gamma_0 - \frac{1}{2} \langle \bar{M}^2 \rangle + \frac{1}{\beta} \left( \frac{\bar{M}^2}{2} + \frac{1}{12} \bar{M}^4 + \ldots \right) \]

\[ = \Gamma_0 + \frac{K_B}{2} \left( T - \frac{2 \pi J}{K_B} \right) \bar{M}^2 + b(T) \bar{M}^4 + \ldots \]

\[ = \bar{M}_0 + a(T) \bar{M}^2 + b(T) \bar{M}^4 + \ldots \]

which is the Landau free energy (expansion in terms of the order parameter \( \bar{M} \)). The coefficients of the Landau theory are
\[ a(T) = \frac{K_B}{2} \left( T - \frac{2 \pi J}{K_B} \right) \]

\[ b(T) = b(T_c) = \frac{K_B T_c}{12} \]

We introduce the critical temperature
\[ T_c = \frac{2 \pi J}{K_B} \]

If we plot \( \Gamma [\bar{M}] \) as a function of \( \bar{M} \) for different temperatures
\[ \Gamma [\bar{M}] \sim \bar{M}^2, \ldots, \bar{M}^4 \quad T > T_c \]
\[ \Gamma [\bar{M}] \sim 0 \quad \bar{M} = 0 \]
\[ \Gamma [\bar{M}] \sim \frac{K_B T_c}{12} \bar{M}^4 \quad \bar{M} = \bar{M}_0 \]
\[ \frac{\delta \bar{\eta}}{\delta \bar{M}} = 0 \Rightarrow \frac{\delta \bar{\eta}}{\delta \bar{N}} = B \mu_B, \quad \text{sc.} \quad \frac{\delta \bar{\eta}}{\delta \bar{M}} = \frac{\delta \bar{\eta}}{\delta \bar{N}} = \frac{B \mu_B}{\mu_B} = B \]

Comme
\[ \bar{\eta}[\bar{N}] = \frac{N[A]}{N} = \frac{a'(T-T_c)N^2}{\alpha(T)} + \frac{k_b T_c M^4}{12} + \ldots \]

Il s'ensuit

1) \[ \frac{\delta \bar{\eta}}{\delta \bar{M}} = B \mu_B = 2 \alpha(T) \bar{M} + 4 b(T_c) \bar{M}^3 \]

À \( T = T_c \), \( \alpha(T) = 0 \) .. \[ \bar{M} = \sqrt{3} \frac{B \mu_B}{4 b(T_c)} = \frac{\beta c m B^{\frac{1}{3}}}{S} \quad S = 3 \]

Isotherme critique

\( S = 3 \) : exposant critique.

2) à \( B = 0 \), \( T \) donnée :
\[ \frac{\delta \bar{\eta}}{\delta \bar{N}} = 0 = 2 \bar{M} \alpha(T) + 4 b(T_c) \bar{M}^3 \]

\[ \bar{M} \left( \alpha(T) + 2 b(T_c) \bar{M}^2 \right) = 0 \Rightarrow \bar{M} = 0 \quad \text{(solution triviale)} \]

ou
\[ \bar{M}^2 \rightarrow \bar{M}_0^2 = - \frac{\alpha(T)}{2 b(T_c)} = \frac{k_b c m (T_c - T)}{12} \]

\( a(T) \neq 0 \) condition nécessaire :

\[ \frac{1}{2} (T_c - T_c)^2 \bar{M}_0 \]

Exp. critique d'.experimentalisation.
Susceptibilité magnétique

3) On revient à

\[ \frac{\partial \bar{M}}{\partial M} = \mu_B \frac{\partial B}{\partial M} = \mu_B \chi^{-1} \]

\[ (\text{Note: } \frac{\partial M}{\partial B}) = \chi \]

\[ \frac{\partial^2 \bar{M}}{\partial M^2} = \mu_B^2 \frac{\partial B}{\partial M} = \mu_B^2 N \chi^{-1} \]

\[ \frac{\partial^2 \bar{M}}{\partial M \partial N} = \frac{\partial}{\partial M} \left[ 2 \alpha(T) \bar{M}^3 + 4 \beta(T) \bar{M} \right] \]

\[ = 2 \alpha(T) + 12 \beta(T) \bar{M}^2 = \mu_B^2 N \chi^{-1} \]

\[ \Rightarrow \chi(T) = \frac{\frac{\mu_B^2 N}{2 \alpha(T) + 12 \beta(T) \bar{M}^2}}{\frac{\mu_B^2 N}{2 \alpha(T)}} = \frac{2 \mu_B N / \kappa_B}{T - T_c} \frac{C}{(T_c - T)^\gamma} \]

\[ \gamma = 1 \]

Exposant critique

\[ \frac{T - T_c}{T_c} = 3 \left( \frac{T_c - T}{T_c} \right) \]

\[ \chi(T \to T_c^-) \]

\[ \frac{\mu_B^2 N}{\kappa_B (T - T_c) + \frac{1}{2} \kappa_B T_c^3 \left( T - T_c \right) \frac{T_c}{2}} \]

\[ = \frac{\mu_B^2 N / \kappa_B}{2 \left( T_c - T \right)} = \frac{C}{(T_c - T)^\gamma} \]

\[ \gamma' = \gamma = 1 \]

1) Chaleur spécifique (Champ nul)

\[ C = \frac{\partial^2 E}{\partial T^2} \bigg|_{B=0} = -T \frac{\partial^2 M}{\partial T^2} \bigg|_{B=0} \]

\[ \bar{M} [\bar{N}_0] = \frac{\kappa_B}{2} \left( T - T_c \right) \cdot \bar{N}_0^2 + \beta(T_c) \bar{M}_0^4 + \bar{P}_o \]

\[ \chi(T) \]
\[ \overline{P}_{eq}(\overline{M}_0) = a(T) \overline{M}_0^2 + b(T)c(T) \overline{M}_0^4 \]

\[ = a(T) \left(-a(T)\right) + b(T)c(T) \cdot \frac{a(T)}{2b(T)c(T)} \cdot \frac{a(T)}{4b^2(T)c(T)} \]

\[ \overline{P}_{eq}(\overline{M}_0) = -\frac{a^2(T)}{4b(T)c(T)} = -\frac{a'^2(T-T_c)^2}{4b(T)c(T)} \]

\[ -T \frac{\partial^2 \overline{P}_{eq}}{\partial T^2} = \frac{CN}{N} = T \frac{a'^2}{2b(T)c(T)} = \frac{Ac}{N} \quad T < T_c \]

\[ T > T_c \]

\[ \frac{\partial C}{\partial T} \]

\[ T_c \]

\[ \overline{P}_{eq}(\overline{M}_0) \]

\[ \frac{Ac}{N} \]

\[ T \]

\[ \frac{C}{N} \sim (T-T_c)^{-\alpha(c)} \]

\[ \alpha' = \alpha = 0 \quad \text{comme exp. de chaleur spécifique} \]

\[ \text{discontinuité en th. 2e champs Mayen.} \]
Spatial correlations

Within mean field theory, the uniform solution $\bar{M}_c = \bar{M}$, corresponds to the Landau theory near $T_c$:

$$M[M] = M^0 - \sum_{\langle ij \rangle} J_{ij} \bar{M}_i \bar{M}_j + \frac{1}{\beta} \sum_i \left[ \frac{1}{2} \bar{M}_i^2 + \frac{1}{12} \bar{M}_i^4 + \ldots \right]$$

Now if we relax the uniform constraint and develop the Gibbs potential $\mathcal{F}$, we get

$$M[M] = \mathcal{F}^0 - \sum_{\langle ij \rangle} J_{ij} \bar{M}_i \bar{M}_j + \frac{1}{\beta} \sum_i \left[ \frac{1}{2} \bar{M}_i^2 + \frac{1}{12} \bar{M}_i^4 + \ldots \right]$$

Since $\bar{M}_i$ is now a function of the site position $i$, it is possible to get the correlation density $\Sigma$ in "mean field" theory and also the momentum dependence of $G(Q)$, the Fourier transform of the correlation function and in turn to get the exponent $\gamma$.

By definition, the correlation function is

$$(G)_{ij} = \left( \frac{\partial \bar{M}_i}{\partial B_j} \right)_{B=0}$$

which is the response of the order parameter at site $i$ due to a variation of the magnetic field at site $j$. $(G)_{ij}$ can then be viewed as a matrix so that

$$(G)^{-1} = \left( \frac{\partial B_j}{\partial \bar{M}_i} \right)_{B=0}$$

Can be related to the Gibbs potential that is

$$B_j = \frac{\partial \mathcal{F}}{\partial \bar{M}_j} = \frac{\partial \mathcal{F}}{\partial M_j}$$

$$\mu_B B_j = B'_j = \frac{\partial \mathcal{F}}{\partial \bar{M}_j} ; \quad \frac{\partial B_j}{\partial \bar{M}_i} = \frac{\partial^2 \mathcal{F}}{\partial \bar{M}_i \partial \bar{M}_j} = \mu_B B'_i B_j.$$
or \[ \frac{1}{\mu_b^2} \frac{\partial^2 \Pi}{\partial N_i \partial N_j} = \partial B_{ij} = (G)_{ij}^{-1} \]

From the expression of the Gibbs potential, we have

\[(G)_{ij}^{-1} = \frac{1}{\mu_b^2} \left\{ -2 J_{ij}^{'} + \frac{1}{\beta} S_{ij} \left[ 1 + \frac{12}{12} \bar{M}_c^2 + \ldots \right]_{B=0} \right\} \]

Going in Fourier space using

\[ G^{-1}(\hat{q}) = \sum_j (G)_{ij}^{-1} e^{i \hat{q} \cdot (\vec{r}_i - \vec{r}_j)} \]

\[ = \frac{1}{\mu_b^2} \left[ -2 J(\hat{q}) + \frac{1}{\beta} \left( 1 + \bar{M}_0^2(T) + \ldots \right) \right] \]

with the Fourier transform of the exchange

\[ J(\hat{q}) = \sum_j J_{ij} e^{i \hat{q} \cdot (\vec{r}_i - \vec{r}_j)} \]

\[ = \sum_j \sum e^{i \hat{q} \cdot (\vec{r}_i - \vec{r}_j)} \]

\[ = \sum_j \left\{ (e^{i q_x d_0} + e^{-i q_x d_0}) + (e^{i q_y d_0} + e^{-i q_y d_0}) \right\} \]

\[ + (e^{i q_x d_0} + e^{-i q_x d_0}) \}

\[ = 2 J \left( \cos q_x d_0 + \cos q_y d_0 + \cos q_z d_0 \right) \]

Note that at \( \hat{q} = 0 \) \( J(\hat{q} = 0) = J \) \( G = 2 J \)

where \( Z \) is the number of nearest neighbors for each site.

We can write:

\[ G(\hat{q}) = \frac{\mu_b^2}{2 (J(0) - J(\hat{q}^2)) + \left( \frac{1}{\beta} - 2 J(0) \right) + \frac{1}{\beta} \bar{M}_0^2} \]

At \( T > T_c \), we have \( \bar{M}_0 = 0 \) and for small (relevant) wave vector \( \hat{q} \) namely \( q d_0 \ll 1 \), we get
\[ J(0) - J'(q) \approx \frac{2J'}{21} (q_x^2 + q_y^2 + q_z^2) d_0^2 = J'd_0^2 q^2 \]

Also
\[ \frac{1}{\beta} - 2J'(0) = k_B T - 2J'(0) = k_B \left( T - \frac{2J'(0)}{k_B} \right) \]

We define the characteristic temperature
\[ 2J'(0) = T_c \]
\[ T_c = \frac{22J'}{k_B} \]

As obtained earlier. Therefore we have at small \( q \) and \( T > T_c \):

\[ G(q) = \frac{\mu B^2 (k_BT_c)^{-1}}{2J'd_0^2 q^2 + \left( \frac{T-T_c}{T_c} \right)} \]

or

\[ G(q) = \frac{\mu B^2 (k_BT_c)^{-1} \xi^{-1}}{1 + \xi^2(T) q^2} \]

Where
\[ \xi(T) = \xi_0 \left( \frac{T-T_c}{T_c} \right)^{-\nu} \]
\[ \nu = \frac{1}{2} \]

\( \xi(T) \) is the correlation length, and \( \nu = \frac{1}{2} \) is the exponent in this mean field approach. \( G(q) \) has therefore a Lorentzian form.

At \( T_c \), \( \xi^2 \rightarrow \infty \) for \( q \) finite, and \( G(q) \) goes as:

\[ G(q) \sim \frac{1}{q^2} \]

Which implies that \( \xi^2 \rightarrow 0 \).
In this small $q$ limit, we can Fourier transform the correlation function

$$(G)_{ij} \rightarrow G(r = |\mathbf{r_i} - \mathbf{r_j}|) = \frac{d^d}{d^d (KT \varepsilon)} \frac{\varepsilon^{-1}}{4\pi^2} \int \frac{d^d q}{(2\pi)^d} \frac{e^{-i\mathbf{q} \cdot \mathbf{r}}}{\varepsilon^2 q^2 + 1}$$

$d = 3$

Here $q_0 \sim \frac{q_0}{\varepsilon}$ is a cut-off, which is chosen consistently with the small $q$ expansion described before. We can rewrite the expression as

$$G(r) = \frac{\mu_B d_0^3 (K \varepsilon)}{4\pi^2} \frac{\varepsilon^{-1}}{\varepsilon^2 q^2 + 1} \text{Im} \int_{q_0}^{+q_0} \frac{q^2 - 1 dq e^{iqr}}{\varepsilon^2 q^2 + 1}$$

in the complex plane, the integral can be done easily.

Using indeed the Cauchy integral formula for the upper half plane we get

$$G(r) = \frac{\mu_B d_0^3 (K \varepsilon)}{4\pi^2} \frac{\varepsilon^{-1}}{\varepsilon^2 q^2 + 1} \text{Im} \left[ \frac{2\pi i \cdot i e^{-r/\sqrt{2\varepsilon}}}{2i} \right]$$
\[ G(r) = \frac{\mu^2 d_o^3 (k_B T_c)^{-1}}{4\pi} \frac{e^{-r/s}}{r} \]

which is consistent with the form:

\[ G(r) = \frac{\mu^2 d_o^3 (k_B T_c)^{-1}}{4\pi} \frac{e^{-r/s}}{r^{d-2+\gamma}} \]

or once again \( \gamma = 0 \). Note that \( G(r) \sim \frac{1}{r} \) for \( r \ll \xi \), which is homogeneous and indicates already "scaling" and self-similarity. This self-similarity property breaks down at longer distances \( r \gg \xi \) when the exponential form \( e^{-r/\xi} \) dominates the decay of \( G(r) \).

At \( T = T_c \), \( \tilde{M}_0^* = \sqrt{3} \left( \frac{T_c - T}{T_c} \right)^{\frac{1}{2}} \) is finite and \( G(\tilde{q}) \) becomes:

\[ G(\tilde{q}) = \frac{\frac{2}{3} \mu (k_B T_c)^{-1}}{\frac{2 \xi_0^2 q^2}{k_B T_c} + \left( \frac{E - T}{T_c} \right) + 3 \left( \frac{T_c - T}{T_c} \right)} \]

\[ = \frac{2(T_c - T)}{T_c} \]

\[ \cdot \frac{1}{2 \epsilon^*} \]

\[ G(\tilde{q}) = \frac{1}{2} \frac{\mu^2 (k_B T_c)^{-1}}{\xi_0^2 q^2 + \epsilon^*} \]

which is also a Lorentzian and then we can define at \( T \to T_c \)

\[ G(\tilde{q}) \sim \frac{1}{q^{2-\gamma'}} \]

with \( \gamma' = \gamma = 0 \). The Fourier transform proceeds similarly.
Here \( (8 \xi^2) = \frac{J' \partial^2 q^2}{k_B T_c} \), and we have

\[
G(r) = \frac{1}{2} M_b^2 d_0^3 (k_B T_c)^{-1} \frac{e^{-r/\xi}}{(\xi_0^2) \frac{4\pi}{r}} - \frac{m_0^2}{r}
\]

where \( \xi \) is the correlation length below \( T_c \)

\[
\xi = \frac{m_0}{M_b} \left( \frac{T - T_c}{T_c} \right)^{-\frac{2}{\nu}}
\]

It follows that \( \nu = \frac{1}{2} \)
Fig. 6.6. Schematic comparison of typical experimental measurements on a Heisenberg ferromagnet (such as EuS) with the predictions of the molecular field theory (cf. Fig. 3.3).

Note that the curve for 1/\kappa_T is shown only for T > T_c.

Fig. 6.7. (a) The plane triangular lattice, and (b) the simple cubic lattice. The fact that both lattices have the same coordination number (q = 6) means that within the mean field approximation they are predicted to have the same critical temperature (cf. (6.1)). This prediction is at odds with the more accurate approximation procedures to be discussed in Chapter 9, and it is currently believed that the critical temperature depends rather strongly upon lattice dimensionality. After Ziman (1964).
Pour un modèle d'Ising :

\[ \hat{H} = -J \sum_{i} \hat{S}_i \cdot \hat{S}_j \]  
(m=1 compasants)

\(*\) l'existence d'un ordre à longue distance à \( T \neq 0 \) est fortement fonction de la dimensionnalité du système et du nombre de composants du paramètre d'ordre \( \beta \).

\[ \begin{array}{c}
\frac{1}{2} \\
\frac{1}{4} \\
\frac{1}{8}
\end{array} \]

* Les exposants C.M. sont exacts à \( d > 4 \)

* Valeur exacte \( \beta = \frac{1}{8} \) : solution d'Ousager à \( d = 2 \)

* Pentes transition de phase à \( d < 1 \) (m=1).

* Note à \( d \to \infty \) (équivalent aux interactions de longue portée) : \( T_c = T_{c,\text{eff}} = K_\text{exact} \), \( \delta, \alpha, \beta, \gamma \ldots \gamma = \frac{1}{2} \), \( \gamma_{\text{crit}} \).

* La dimension 4 pour une transition ferromagnétique est critique.

* Sont appelées dimension critique supérieure dimension 2 : dimension critique inférieure.

* Pour C.M. (à \( d \leq 2 \) non définie !)