Two dimensional quantum memories

Commuting projector codes

David Poulin

Département de Physique
Université de Sherbrooke

Joint work with Sergey Bravyi and Barbara Terhal

Sydney Quantum information Theory Workshop
Coogee, Australia, January 2013
2D Commuting Projector Codes

Holographic Disentangling Lemma

Holographic Minimum Distance

Capacity-Stability Tradeoff

String-Like Logical Operators

Open Questions
Definitions

- $\Lambda$ is a 2D lattice.
- Each vertex occupied by $d$-level quantum particle.
- Hamiltonian $H = -\sum_{X \subset \Lambda} P_X$ with
  - $P_X = 0$ if radius($X$) $\geq w$.
  - $[P_X, P_Y] = 0$.
  - $P_X$ are projectors (optional).
- Code $\mathcal{C} = \{\psi : P_X|\psi\rangle = |\psi\rangle\}$
  - $= \text{ground space of } H$
  - $= \text{image of code projector } \Pi = \prod_X P_X$
- With proper coarse graining, we can assume that
  - $\Lambda$ is a regular square lattice.
  - Each $P_X$ acts on $2 \times 2$ cell.
Definitions

- \( \Lambda \) is a 2D lattice.
- Each vertex occupied by \( d \)-level quantum particle.
- Hamiltonian \( H = - \sum_{X \subset \Lambda} P_X \) with
  - \( P_X = 0 \) if radius\((X) \geq w \).
  - \([P_X, P_Y] = 0\).
  - \( P_X \) are projectors (optional).
- Code \( \mathcal{C} = \{ \psi : P_X|\psi\rangle = |\psi\rangle \} \)
  - = ground space of \( H \)
  - = image of code projector \( \Pi = \prod_X P_X \)
- With proper coarse graining, we can assume that
  - \( \Lambda \) is a regular square lattice.
  - Each \( P_X \) acts on \( 2 \times 2 \) cell.
Definitions

• Λ is a 2D lattice.
• Each vertex occupied by \( d \)-level quantum particle.
• Hamiltonian \( H = -\sum_{X \subseteq \Lambda} P_X \) with
  • \( P_X = 0 \) if \( \text{radius}(X) \geq w \).
  • \([P_X, P_Y] = 0\).
  • \( P_X \) are projectors (optional).
• Code \( C = \{\psi : P_X|\psi\rangle = |\psi\rangle\} \)
  = ground space of \( H \)
  = image of code projector \( \Pi = \prod_X P_X \)
• With proper coarse graining, we can assume that
  • \( \Lambda \) is a regular square lattice.
  • Each \( P_X \) acts on \( 2 \times 2 \) cell.
Definitions

- \( \Lambda \) is a 2D lattice.
- Each vertex occupied by \( d \)-level quantum particle.
- Hamiltonian \( H = -\sum_{X \subseteq \Lambda} P_X \) with
  - \( P_X = 0 \) if \( \text{radius}(X) \geq w \).
  - \( [P_X, P_Y] = 0 \).
  - \( P_X \) are projectors (optional).
- Code \( \mathcal{C} = \{ \psi : P_X |\psi\rangle = |\psi\rangle \} \)
  - = ground space of \( H \)
  - = image of code projector \( \Pi = \prod_X P_X \)
- With proper coarse graining, we can assume that
  - \( \Lambda \) is a regular square lattice.
  - Each \( P_X \) acts on \( 2 \times 2 \) cell.


Definitions

- $\Lambda$ is a 2D lattice.
- Each vertex occupied by $d$-level quantum particle.
- Hamiltonian $H = -\sum_{X \subseteq \Lambda} P_X$ with
  - $P_X = 0$ if $\text{radius}(X) \geq w$.
  - $[P_X, P_Y] = 0$.
  - $P_X$ are projectors (optional).
- Code $C = \{ \psi : P_X |\psi\rangle = |\psi\rangle \} = \text{ground space of } H$
  - $\text{image of code projector } \Pi = \prod_X P_X$
- With proper coarse graining, we can assume that
  - $\Lambda$ is a regular square lattice.
  - Each $P_X$ acts on $2 \times 2$ cell.
2D Commuting Projector Codes

Definitions

- \( \Lambda \) is a 2D lattice.
- Each vertex occupied by \( d \)-level quantum particle.
- Hamiltonian \( H = - \sum_{X \subset \Lambda} P_X \) with
  - \( P_X = 0 \) if radius\((X) \geq w \).
  - \([P_X, P_Y] = 0 \).
  - \( P_X \) are projectors (optional).
- Code \( \mathcal{C} = \{ \psi : P_X |\psi\rangle = |\psi\rangle \} \)
  = ground space of \( H \)
  = image of code projector \( \Pi = \prod_X P_X \)
- With proper coarse graining, we can assume that
  - \( \Lambda \) is a regular square lattice.
  - Each \( P_X \) acts on \( 2 \times 2 \) cell.
Definitions

- $\Lambda$ is a 2D lattice.
- Each vertex occupied by $d$-level quantum particle.
- Hamiltonian $H = -\sum_{X \subseteq \Lambda} P_X$ with
  - $P_X = 0$ if radius$(X) \geq w$.
  - $[P_X, P_Y] = 0$.
  - $P_X$ are projectors (optional).
- Code $C = \{ \psi : P_X |\psi\rangle = |\psi\rangle \}$
  = ground space of $H$
  = image of code projector $\Pi = \prod_X P_X$
- With proper coarse graining, we can assume that
  - $\Lambda$ is a regular square lattice.
  - Each $P_X$ acts on $2 \times 2$ cell.
Definitions

- $\Lambda$ is a 2D lattice.
- Each vertex occupied by $d$-level quantum particle.
- Hamiltonian $H = -\sum_{X \subseteq \Lambda} P_X$ with
  - $P_X = 0$ if $\text{radius}(X) \geq w$.
  - $[P_X, P_Y] = 0$.
  - $P_X$ are projectors (optional).
- Code $C = \{ \psi : P_X |\psi\rangle = |\psi\rangle \}$
  - $|\psi\rangle$ = ground space of $H$
  - $|\psi\rangle$ = image of code projector $\Pi = \prod_X P_X$
- With proper coarse graining, we can assume that
  - $\Lambda$ is a regular square lattice.
  - Each $P_X$ acts on $2 \times 2$ cell.
Definitions

- $\Lambda$ is a 2D lattice.
- Each vertex occupied by $d$-level quantum particle.
- Hamiltonian $H = -\sum_{X \subset \Lambda} P_X$ with
  - $P_X = 0$ if radius($X$) $\geq w$.
  - $[P_X, P_Y] = 0$.
  - $P_X$ are projectors (optional).
- Code $C = \{\psi : P_X|\psi\rangle = |\psi\rangle\}$
  - = ground space of $H$
  - = image of code projector $\Pi = \prod_X P_X$
- With proper coarse graining, we can assume that
  - $\Lambda$ is a regular square lattice.
  - Each $P_X$ acts on $2 \times 2$ cell.
Definitions

- $\Lambda$ is a 2D lattice.
- Each vertex occupied by $d$-level quantum particle.
- Hamiltonian $H = -\sum_{X \subset \Lambda} P_X$ with
  - $P_X = 0$ if radius($X$) $\geq w$.
  - $[P_X, P_Y] = 0$.
  - $P_X$ are projectors (optional).
- Code $C = \{\psi : P_X|\psi\rangle = |\psi\rangle\}$
  - = ground space of $H$
  - = image of code projector $\Pi = \prod_X P_X$
- With proper coarse graining, we can assume that
  - $\Lambda$ is a regular square lattice.
  - Each $P_X$ acts on $2 \times 2$ cell.
Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
- Levin & Wen’s string-net models
- Turaev-Viro models
- Kitaev’s quantum double models
- Most known models with topological quantum order

Remark
The first two examples are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark
Subsystem codes do not belong to this family.
Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
  - Levin & Wen’s string-net models
  - Turaev-Viro models
  - Kitaev’s quantum double models
  - Most known models with topological quantum order

Remark
The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark
Subsystem codes do not belong to this family.
Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
- Levin & Wen’s string-net models
  - Turaev-Viro models
  - Kitaev’s quantum double models
  - Most known models with topological quantum order

Remark
The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark
Subsystem codes do not belong to this family.
Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
- Levin & Wen’s string-net models
- Turaev-Viro models
  - Kitaev’s quantum double models
  - Most known models with topological quantum order

Remark
The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark
Subsystem codes do not belong to this family.
Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
- Levin & Wen’s string-net models
- Turaev-Viro models
- Kitaev’s quantum double models

Most known models with topological quantum order

Remark
The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark
Subsystem codes do not belong to this family.
Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
- Levin & Wen’s string-net models
- Turaev-Viro models
- Kitaev’s quantum double models
- Most known models with topological quantum order

Remark

The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark

Subsystem codes do not belong to this family.
Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
- Levin & Wen’s string-net models
- Turaev-Viro models
- Kitaev’s quantum double models
- Most known models with topological quantum order

Remark
The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark
Subsystem codes do not belong to this family.
Well known examples

- Kitaev’s toric code
- Bombin’s topological color codes
- Levin & Wen’s string-net models
- Turaev-Viro models
- Kitaev’s quantum double models
- Most known models with topological quantum order

Remark

The first two example are simple because they are stabilizer codes. Most things I will say are trivial to prove in this case.

Remark

Subsystem codes do not belong to this family.
Correctable region

A region $M \subset \Lambda$ is **correctable** if there exists a recovery operation $\mathcal{R}$ such that $\mathcal{R}(\text{Tr}_M \rho) = \rho$ for all code states $\rho$.

Minimum distance

The minimum distance $d$ is the size of the smallest non-correctable region.

Logical operator

Operator $L$ such that $L|\psi\rangle$ is a code state for any code state $|\psi\rangle$.

Rate (capacity)

The rate of a code is $R = \frac{k}{n}$ where $k = \log \dim(C)$ and $n = |\Lambda|$ in the number of particles.
Correctable region

A region $M \subset \Lambda$ is *correctable* if there exists a recovery operation $R$ such that $R(\text{Tr}_M \rho) = \rho$ for all code states $\rho$.

Minimum distance

The minimum distance $d$ is the size of the smallest non-correctable region.

Logical operator

Operator $L$ such that $L|\psi\rangle$ is a code state for any code state $|\psi\rangle$.

Rate (capacity)

The rate of a code is $R = \frac{k}{n}$ where $k = \log \dim(C)$ and $n = |\Lambda|$ in the number of particles.
**Standard definitions**

**Correctable region**

A region \( M \subset \Lambda \) is *correctable* if there exists a recovery operation \( R \) such that \( R(\text{Tr}_M \rho) = \rho \) for all code states \( \rho \).

**Minimum distance**

The minimum distance \( d \) is the size of the smallest non-correctable region.

**Logical operator**

Operator \( L \) such that \( L|\psi\rangle \) is a code state for any code state \( |\psi\rangle \).

**Rate (capacity)**

The rate of a code is \( R = \frac{k}{n} \) where \( k = \log \dim(\mathcal{C}) \) and \( n = |\Lambda| \) in the number of particles.
### Correctable region

A region \( M \subset \Lambda \) is *correctable* if there exists a recovery operation \( \mathcal{R} \) such that \( \mathcal{R}(\text{Tr}_M \rho) = \rho \) for all code states \( \rho \).

### Minimum distance

The minimum distance \( d \) is the size of the smallest non-correctable region.

### Logical operator

Operator \( L \) such that \( L |\psi\rangle \) is a code state for any code state \( |\psi\rangle \).

### Rate (capacity)

The rate of a code is \( R = \frac{k}{n} \) where \( k = \log \dim(C) \) and \( n = |\Lambda| \) in the number of particles.
Outline

1. 2D Commuting Projector Codes
2. Holographic Disentangling Lemma
3. Holographic Minimum Distance
4. Capacity-Stability Tradeoff
5. String-Like Logical Operators
6. Open Questions
Holographic disentangling lemma

Let $M \subset \Lambda$ be a correctable region and suppose that its boundary $\partial M$ is also correctable. Then, there exists a unitary operator $U_{\partial M}$ acting only on the boundary of $M$ such that, for any code state $|\psi\rangle$,

$$U_{\partial M}|\psi\rangle = |\phi_M\rangle \otimes |\psi'_M\rangle$$

for some fixed state $|\phi_M\rangle$ on $M$.

Remark

For a trivial code $k = 0$, every region is correctable, so we recover the area law $S(M) \leq |\partial M|$ for commuting Hamiltonians of Wolf, Verstraete, Hastings, and Cirac.
Holographic Disentangling Lemma

Statement of the lemma

Holographic disentangling lemma

Let $M \subset \Lambda$ be a correctable region and suppose that its boundary $\partial M$ is also correctable. Then, there exists a unitary operator $U_{\partial M}$ acting only on the boundary of $M$ such that, for any code state $|\psi\rangle$,

$$U_{\partial M} |\psi\rangle = |\phi_M\rangle \otimes |\psi'_{\bar{M}}\rangle$$

for some fixed state $|\phi_M\rangle$ on $M$.

Remark

For a trivial code $k = 0$, every region is correctable, so we recover the area law $S(M) \leq |\partial M|$ for commuting Hamiltonians of Wolf, Verstraete, Hastings, and Cirac.
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB}P_{B\overline{M}}$ with $[P_{AB}, P_{B\overline{M}}] = 0$.

$$
\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B^J_L} \otimes \mathcal{H}_{B^J_R}
$$
and

$$
\Pi = \bigoplus_J P_{AB^J_L} \otimes P_{B^J_R\overline{M}}
$$

This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.

- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that

$$
\Pi = V_B P_{AB^1} \otimes P_{B^2\overline{M}} V_B^\dagger. \quad (\star)
$$
Let $M$ be correctable.
Assume $\partial M$ is correctable.

Let $M = A \cup B$, $\bar{M} = C \cup D$, and $\partial M = B \cup C$.

Write $\Pi = P_{AB}P_{B\bar{M}}$ with $[P_{AB}, P_{B\bar{M}}] = 0$.

$\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B^J_L} \otimes \mathcal{H}_{B^J_R}$ and $\Pi = \bigoplus_J P_{AB^J_L} \otimes P_{B^J_R\bar{M}}$.

This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.

We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2\bar{M}} V_B^\dagger$. (⋆)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\bar{M} = C \cup D$, and $\partial M = B \cup C$.

Write $\Pi = P_{AB}P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.

$\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B_J^L} \otimes \mathcal{H}_{B_J^R}$ and $\Pi = \bigoplus_J P_{AB_J^L} \otimes P_{B_J^R \bar{M}}$

This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.

We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2 \bar{M}} V_B^\dagger$. (*)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.

Write $\Pi = P_{AB}P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.

- $\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B^J_L} \otimes \mathcal{H}_{B^J_R}$ and $\Pi = \bigoplus_J P_{AB^J_L} \otimes P_{B^J_R} \overline{M}$
- This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.
- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2} \overline{M} V_B^\dagger$. (⋆)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB}P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.

\[ H_B = \bigoplus_J H_{B_L^J} \otimes H_{B_R^J} \] and $\Pi = \bigoplus_J P_{AB_L^J} \otimes P_{B_R^J \overline{M}}$

This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.

- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2 \overline{M}} V_B^\dagger$. ($\star$)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB} P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.

\[ \mathcal{H}_B = \bigoplus_J \mathcal{H}_{B^J_L} \otimes \mathcal{H}_{B^J_R} \quad \text{and} \quad \Pi = \bigoplus_J P_{AB^J_L} \otimes P_{B^J_R \overline{M}} \]

- This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.
- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2 \overline{M}} V_B^\dagger$. ($\star$)
Holographic Disentangling Lemma

Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB} P_{B\overline{M}}$ with $[P_{AB}, P_{B\overline{M}}] = 0$.
- $\mathcal{H}_B = \bigoplus_J \mathcal{H}_{B_L}^J \otimes \mathcal{H}_{B_R}^J$ and $\Pi = \bigoplus_J P_{AB_L}^J \otimes P_{B_R}^J \overline{M}$
- This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.
- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB^1} \otimes P_{B^2 \overline{M}} V_B^\dagger$. (⋆)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\bar{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{AB}P_{BM}$ with $[P_{AB}, P_{BM}] = 0$.

\[ \mathcal{H}_B = \bigoplus_J \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R} \] and $\Pi = \bigoplus_J P_{AB_L} \otimes P_{B_R M}$

This last sum over $J$ contains only one non-zero factor since $B \subset M$ is correctable.

- We can divide $B$ into two subsystems $B^1$ and $B^2$ such that $\Pi = V_B P_{AB} \otimes P_{B^2 \bar{M}} V_B^\dagger$. (⋆)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\bar{M} = C \cup D$, and $\partial M = B \cup C$.
  Write $\Pi = P_{MC}P_{CD}$ with $[P_{MC}, P_{CD}] = 0$.

$\mathcal{H}_C = \bigoplus_J \mathcal{H}_{C_L^J} \otimes \mathcal{H}_{C_R^J}$ and $\Pi = \bigoplus_J P_{MC_L^J} \otimes P_{C_R^J D}$

This last sum over $J$ contains only one non-zero factor since $C \subset \partial M$ is correctable.

We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC} P_{C^2 D} V_C^\dagger$. (***)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{MC}P_{CD}$ with $[P_{MC}, P_{CD}] = 0$.

- $\mathcal{H}_C = \bigoplus_J \mathcal{H}_{C_L} \otimes \mathcal{H}_{C_R}$ and $\Pi = \bigoplus_J P_{MC_L} \otimes P_{C_R}$
- This last sum over $J$ contains only one non-zero factor since $C \subset \partial M$ is correctable.
- We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC^1} \otimes P_{C^2 D} V_C^\dagger$. (***)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{MC}P_{CD}$ with $[P_{MC}, P_{CD}] = 0$.

$$\mathcal{H}_C = \bigoplus_J \mathcal{H}_{C^J_L} \otimes \mathcal{H}_{C^J_R}$$ and $\Pi = \bigoplus_J P_{MC^J_L} \otimes P_{C^J_R D}$

This last sum over $J$ contains only one non-zero factor since $C \subset \partial M$ is correctable.

- We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC^1} \otimes P_{C^2 D} V_C^\dagger$. (***)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{MC}P_{CD}$ with $[P_{MC}, P_{CD}] = 0$.

- $\mathcal{H}_C = \bigoplus_J \mathcal{H}_{C_L}^J \otimes \mathcal{H}_{C_R}^J$ and $\Pi = \bigoplus_J P_{MC_L}^J \otimes P_{C_R}^J D$

- This last sum over $J$ contains only one non-zero factor since $C \subset \partial M$ is correctable.

- We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC^1} \otimes P_{C^2 D} V_C^\dagger$. (***)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.
- Write $\Pi = P_{MC} P_{CD}$ with $[P_{MC}, P_{CD}] = 0$.

\[ \mathcal{H}_C = \bigoplus_J \mathcal{H}_{C_L^J} \otimes \mathcal{H}_{C_R^J} \text{ and } \Pi = \bigoplus_J P_{MC_L^J} \otimes P_{C_R^J D} \]

This last sum over $J$ contains only one non-zero factor since $C \subset \partial M$ is correctable.

- We can divide $C$ into two subsystems $C^1$ and $C^2$ such that $\Pi = V_C P_{MC^1} \otimes P_{C^2 D} V_C^\dagger$. (***)
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.

Combining ($\star$) with ($\star\star$), $\Pi' = V_B^\dagger V_C^\dagger \cap V_B V_C = P_{AB^1} P_{B^2C^1} P_{C^2D}$
- $P_{AB^1} = |\eta_{AB^1}\rangle \langle \eta_{AB^1}|$ is rank one since $AB^1 \subset M$ is correctable.
- $P_{B^2C^1} = |\nu_{B^2C^1}\rangle \langle \nu_{B^2C^1}|$ is rank one since $B^2C^1 \subset \partial M$ is correctable.
- Let $V_{B^2C^1}$ be any unitary such that $V_{B^2C^1} |\nu_{B^2C^1}\rangle = |\alpha_{B^2}\rangle \otimes |\beta_{C^2}\rangle$.
- Then $U_{\partial M} = V_{B^2C^1} V_B^\dagger V_C^\dagger$ disentangles region $M$ as claimed.
Holographic Disentangling Lemma

Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.

Combining ($\star$) with ($\star\star$), $\Pi' = V_B^\dagger V_C^\dagger \Pi V_B V_C = P_{AB^1} P_{B^2C^1} P_{C^2D}$

- $P_{AB^1} = |\eta_{AB^1}\rangle \langle \eta_{AB^1}|$ is rank one since $AB^1 \subset M$ is correctable.
- $P_{B^2C^1} = |\nu_{B^2C^1}\rangle \langle \nu_{B^2C^1}|$ is rank one since $B^2C^1 \subset \partial M$ is correctable.
- Let $V_{B^2C^1}$ be any unitary such that $V_{B^2C^1} |\nu_{B^2C^1}\rangle = |\alpha_{B^2}\rangle \otimes |\beta_{C^2}\rangle$.
- Then $U_{\partial M} = V_{B^2C^1} V_B^\dagger V_C^\dagger$ disentangles region $M$ as claimed.
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.

Combining ($\star$) with ($\star\star$), $\Pi' = V_B^\dagger V_C^\dagger \prod V_B V_C = P_{AB^1} P_{B^2 C^1} P_{C^2 D}$

- $P_{AB^1} = |\eta_{AB^1}\rangle\langle\eta_{AB^1}|$ is rank one since $AB^1 \subset M$ is correctable.
- $P_{B^2 C^1} = |\nu_{B^2 C^1}\rangle\langle\nu_{B^2 C^1}|$ is rank one since $B^2 C^1 \subset \partial M$ is correctable.
- Let $V_{B^2 C^1}$ be any unitary such that $V_{B^2 C^1} |\nu_{B^2 C^1}\rangle = |\alpha_{B^2}\rangle \otimes |\beta_{C^2}\rangle$.
- Then $U_{\partial M} = V_{B^2 C^1} V_B^\dagger V_C^\dagger$ disentangles region $M$ as claimed.
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\overline{M} = C \cup D$, and $\partial M = B \cup C$.

Combining (⋆) with (⋆⋆), $\Pi' = V_B^\dagger V_C^\dagger \Pi V_B V_C = P_{AB^1} P_{B^2C^1} P_{C^2D}$

- $P_{AB^1} = |\eta_{AB^1}\rangle \langle \eta_{AB^1}|$ is rank one since $AB^1 \subset M$ is correctable.
- $P_{B^2C^1} = |\nu_{B^2C^1}\rangle \langle \nu_{B^2C^1}|$ is rank one since $B^2C^1 \subset \partial M$ is correctable.

Let $V_{B^2C^1}$ be any unitary such that $V_{B^2C^1}|\nu_{B^2C^1}\rangle = |\alpha_{B^2}\rangle \otimes |\beta_{C^2}\rangle$.

Then $U_{\partial M} = V_{B^2C^1} V_B^\dagger V_C^\dagger$ disentangles region $M$ as claimed.
Proof

- Let $M$ be correctable.
- Assume $\partial M$ is correctable.
- Let $M = A \cup B$, $\bar{M} = C \cup D$, and $\partial M = B \cup C$.

Combining ($\star$) with ($\star \star$), $\Pi' = V_B^\dagger V_C^\dagger \cap V_B V_C = P_{AB^1} P_{B^2 C^1} P_{C^2 D}$

$P_{AB^1} = |\eta_{AB^1}\rangle \langle \eta_{AB^1}|$ is rank one since $AB^1 \subset M$ is correctable.

$P_{B^2 C^1} = |\nu_{B^2 C^1}\rangle \langle \nu_{B^2 C^1}|$ is rank one since $B^2 C^1 \subset \partial M$ is correctable.

Let $V_{B^2 C^1}$ be any unitary such that $V_{B^2 C^1} |\nu_{B^2 C^1}\rangle = |\alpha_{B^2}\rangle \otimes |\beta_{C^2}\rangle$.

Then $U_{\partial M} = V_{B^2 C^1} V_B^\dagger V_C^\dagger$ disentangles region $M$ as claimed.
Let \( M \) be correctable.

Assume \( \partial M \) is correctable.

Let \( M = A \cup B, \overline{M} = C \cup D, \) and \( \partial M = B \cup C. \)

Combining \((\ast)\) with \((\ast\ast)\), \( \Pi' = V_B^\dagger V_C^\dagger \Pi V_B V_C = P_{AB^1} P_{B^2C^1} P_{C^2D} \)

\[ P_{AB^1} = |\eta_{AB^1}\rangle \langle \eta_{AB^1}| \text{ is rank one since } AB^1 \subset M \text{ is correctable.} \]

\[ P_{B^2C^1} = |\nu_{B^2C^1}\rangle \langle \nu_{B^2C^1}| \text{ is rank one since } B^2C^1 \subset \partial M \text{ is correctable.} \]

Let \( V_{B^2C^1} \) be any unitary such that \( V_{B^2C^1} |\nu_{B^2C^1}\rangle = |\alpha_{B^2}\rangle \otimes |\beta_{C^2}\rangle. \)

Then \( U_{\partial M} = V_{B^2C^1} V_B^\dagger V_C^\dagger \) disentangles region \( M \) as claimed.
Outline

1. 2D Commuting Projector Codes
2. Holographic Disentangling Lemma
3. Holographic Minimum Distance
4. Capacity-Stability Tradeoff
5. String-Like Logical Operators
6. Open Questions
Holographic Minimum Distance

Statement of the result

Holographic minimum distance

Region $M \subset \Lambda$ is correctable if its boundary is smaller than the minimum distance $|\partial M| \leq cd$.

- Bulky errors are not problematic: it’s the skinny ones we need to worry about.
- This hints at our next result: string-like logical operators.

David Poulin (Sherbrooke)

2D Quantum Memories

Coogee’13
Holographic minimum distance

Region $M \subset \Lambda$ is correctable if its boundary is smaller than the minimum distance $|\partial M| \leq cd$.

- Bulky errors are not problematic: it’s the skinny ones we need to worry about.
- This hints at our next result: string-like logical operators.
Statement of the result

Holographic minimum distance

Region \( M \subset \Lambda \) is correctable if its boundary is smaller than the minimum distance \( |\partial M| \leq cd \).

- Bulky errors are not problematic: it’s the skinny ones we need to worry about.
- This hints at our next result: string-like logical operators.
Let $M \subset \Lambda$ be a correctable region.

If $|\partial M| \leq d$, then $\partial M$ is also correctable.

Thus, we can reconstruct any code state $\rho$ from $\rho_{AD} = \text{Tr}_{\partial M} \rho$.

But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with $\eta_A$ independent of the encoded state $\rho$.

Thus, we can reconstruct $\rho$ from $\rho_D = \text{Tr}_{M \cup \partial M} \rho$, so $M \cup \partial M$ is correctable.

We can continue to grow $M$ this way until $|\partial M| \geq d$. 
Proof

- Let $M \subset \Lambda$ be a correctable region.
- If $|\partial M| \leq d$, then $\partial M$ is also correctable.
- Thus, we can reconstruct any code state $\rho$ from $\rho_{AD} = \text{Tr}_{\partial M} \rho$.
- But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with $\eta_A$ independent of the encoded state $\rho$.
- Thus, we can reconstruct $\rho$ from $\rho_D = \text{Tr}_{M \cup \partial M} \rho$, so $M \cup \partial M$ is correctable.
- We can continue to grow $M$ this way until $|\partial M| \geq d$. 

$\tilde{M} = \Lambda \backslash M$
Proof

- Let \( M \subset \Lambda \) be a correctable region.
- If \(|\partial M| \leq d\), then \( \partial M \) is also correctable.
- Thus, we can reconstruct any code state \( \rho \) from \( \rho_{AD} = \text{Tr}_{\partial M} \rho \).
- But from the Holographic disentangling lemma, \( \rho_{AD} = \eta_A \otimes \rho_D \) with \( \eta_A \) independent of the encoded state \( \rho \).
- Thus, we can reconstruct \( \rho \) from \( \rho_D = \text{Tr}_{M \cup \partial M} \rho \), so \( M \cup \partial M \) is correctable.
- We can continue to grow \( M \) this way until \(|\partial M| \geq d\).
Proof

- Let $M \subset \Lambda$ be a correctable region.
- If $|\partial M| \leq d$, then $\partial M$ is also correctable.
- Thus, we can reconstruct any code state $\rho$ from $\rho_{AD} = \text{Tr}_{\partial M} \rho$.
- But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with $\eta_A$ independent of the encoded state $\rho$.
- Thus, we can reconstruct $\rho$ from $\rho_D = \text{Tr}_{M \cup \partial M} \rho$, so $M \cup \partial M$ is correctable.
- We can continue to grow $M$ this way until $|\partial M| \geq d$. 
Proof

- Let $M \subset \Lambda$ be a correctable region.
- If $|\partial M| \leq d$, then $\partial M$ is also correctable.
- Thus, we can reconstruct any code state $\rho$ from $\rho_{AD} = \text{Tr}_{\partial M} \rho$.
- But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with $\eta_A$ independent of the encoded state $\rho$.
- Thus, we can reconstruct $\rho$ from $\rho_D = \text{Tr}_{M \cup \partial M} \rho$, so $M \cup \partial M$ is correctable.
- We can continue to grow $M$ this way until $|\partial M| \geq d$. 
Proof

- Let $M \subset \Lambda$ be a correctable region.
- If $|\partial M| \leq d$, then $\partial M$ is also correctable.
- Thus, we can reconstruct any code state $\rho$ from $\rho_{AD} = \text{Tr}_{\partial M} \rho$.
- But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with $\eta_A$ independent of the encoded state $\rho$.
- Thus, we can reconstruct $\rho$ from $\rho_D = \text{Tr}_{M \cup \partial M} \rho$, so $M \cup \partial M$ is correctable.
- We can continue to grow $M$ this way until $|\partial M| \geq d$. 
Proof

- Let $M \subset \Lambda$ be a correctable region.
- If $|\partial M| \leq d$, then $\partial M$ is also correctable.
- Thus, we can reconstruct any code state $\rho$ from $\rho_{AD} = \text{Tr}_{\partial M} \rho$.
- But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with $\eta_A$ independent of the encoded state $\rho$.
- Thus, we can reconstruct $\rho$ from $\rho_D = \text{Tr}_{M \cup \partial M} \rho$, so $M \cup \partial M$ is correctable.
- We can continue to grow $M$ this way until $|\partial M| \geq d$. 
Let $M \subset \Lambda$ be a correctable region.

If $|\partial M| \leq d$, then $\partial M$ is also correctable.

Thus, we can reconstruct any code state $\rho$ from $\rho_{AD} = \text{Tr}_{\partial M} \rho$.

But from the Holographic disentangling lemma, $\rho_{AD} = \eta_A \otimes \rho_D$ with $\eta_A$ independent of the encoded state $\rho$.

Thus, we can reconstruct $\rho$ from $\rho_D = \text{Tr}_{M \cup \partial M} \rho$, so $M \cup \partial M$ is correctable.

We can continue to grow $M$ this way until $|\partial M| \geq d$. 
Outline

1. 2D Commuting Projector Codes
2. Holographic Disentangling Lemma
3. Holographic Minimum Distance
4. Capacity-Stability Tradeoff
5. String-Like Logical Operators
6. Open Questions
Statement of the result

Capacity-Stability Tradeoff

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2^n} \log 3 - H\left( \frac{d}{2^n} \right) \right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Statement of the result

**Capacity-Stability Tradeoff**

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left( \frac{d}{2n} \right) \right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
### Statement of the result

**Capacity-Stability Tradeoff**

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[1 - \frac{d}{2n} \log 3 - H\left(\frac{d}{2n}\right)\right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Statement of the result

Capacity-Stability Tradeoff

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left(\frac{d}{2n}\right) \right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto \frac{n}{\ell^2} \propto \frac{n}{d^2} \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Statement of the result

**Capacity-Stability Tradeoff**

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left(\frac{d}{2n}\right) \right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Statement of the result

Capacity-Stability Tradeoff

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left( \frac{d}{2n} \right) \right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Statement of the result

Capacity-Stability Tradeoff

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left( \frac{d}{2n} \right) \right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).

- No “good codes" in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Statement of the result

Capacity-Stability Tradeoff

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left(\frac{d}{2n}\right) \right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Statement of the result

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \[ k \leq n - 2(d - 1) \].
- Hamming bound: \[ k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left( \frac{d}{2n} \right) \right] \].
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto \frac{n}{\ell^2} \propto \frac{n}{d^2} \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Capacity-Stability Tradeoff

\[ k \leq c \frac{n}{d^2} \]

Singleton’s bound: \( k \leq n - 2(d - 1) \).

Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left(\frac{d}{2n}\right) \right] \).

Kitaev’s codes (with punctures) saturate this bound, so it is tight.
- Holes of linear size \( \ell \) separated by distance \( \ell \).
- Minimum distance \( d \propto \ell \).
- Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).

No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).

For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Statement of the result

\[ k \leq c \frac{n}{d^2} \]

- Singleton’s bound: \( k \leq n - 2(d - 1) \).
- Hamming bound: \( k \leq n \left[ 1 - \frac{d}{2n} \log 3 - H\left(\frac{d}{2n}\right) \right] \).
- Kitaev’s codes (with punctures) saturate this bound, so it is tight.
  - Holes of linear size \( \ell \) separated by distance \( \ell \).
  - Minimum distance \( d \propto \ell \).
  - Number of logical qubits \( k \propto \) number of holes \( \propto n/\ell^2 \propto n/d^2 \).
- No “good codes” in 2D, i.e. \( k \propto n \) and \( d \propto n \).
- For classical codes, \( k \leq c \frac{n}{\sqrt{d}} \).

We will need two tools to prove this result.
Information-theoretic condition for error correction

$M$ is correctable iff $S(MM) = S(M) - S(M)$ for any code state $\rho$.

- Obvious for pure states.
- Let $\rho_{MM}$ be a code state and $\rho_{MMR}$ its purification.
- By assumption, there exists $\mathcal{R}$ on $\Lambda$ such that
  $\mathcal{R}(\text{Tr}_M\rho_{MMR}) = \rho_{MMR}$ and $\mathcal{R}(\text{Tr}_M\rho_{MM} \otimes \rho_R) = \rho_{MM} \otimes \rho_R$.
- Since relative entropy can only decrease under the action of a CPTP map,
  $S(\rho_{MMR} \parallel \rho_{MM} \otimes \rho_R) = S(\text{Tr}_M\rho_{MMR} \parallel \text{Tr}_M\rho_{MM} \otimes \rho_R)$.
- Using $S(\rho_{AB} \parallel \rho_A \otimes \rho_B) = S(A) + S(B) - S(AB)$ and the fact that
  $\rho_{MMR}$ is pure, we get the desired result.
Information-theoretic condition for error correction

\( M \) is correctable iff \( S(\overline{M}) = S(\overline{M}) - S(\overline{M}) \) for any code state \( \rho \).

- Obvious for pure states.
- Let \( \rho_{\overline{MM}} \) be a code state and \( \rho_{\overline{MMR}} \) its purification.
- By assumption, there exists \( \mathcal{R} \) on \( \Lambda \) such that
  \( \mathcal{R}(\text{Tr}_M \rho_{\overline{MMR}}) = \rho_{\overline{MMR}} \) and \( \mathcal{R}(\text{Tr}_M \rho_{\overline{MM}} \otimes \rho_R) = \rho_{\overline{MM}} \otimes \rho_R \).
- Since relative entropy can only decrease under the action of a CPTP map,
  \( S(\rho_{\overline{MMR}} \parallel \rho_{\overline{MM}} \otimes \rho_R) = S(\text{Tr}_M \rho_{\overline{MMR}} \parallel \text{Tr}_M \rho_{\overline{MM}} \otimes \rho_R) \).
- Using \( S(\rho_{AB} \parallel \rho_A \otimes \rho_B) = S(A) + S(B) - S(AB) \) and the fact that \( \rho_{\overline{MMR}} \) is pure, we get the desired result.
Information-theoretic condition for error correction

\( M \) is correctable iff \( S(\overline{MM}) = S(\overline{M}) - S(M) \) for any code state \( \rho \).

- Obvious for pure states.
- Let \( \rho_{\overline{MM}} \) be a code state and \( \rho_{\overline{M}M\overline{R}} \) its purification.
  - By assumption, there exists \( \mathcal{R} \) on \( \Lambda \) such that
    \( \mathcal{R}(\text{Tr}_M \rho_{\overline{M}M\overline{R}}) = \rho_{\overline{M}M\overline{R}} \) and \( \mathcal{R}(\text{Tr}_M \rho_{\overline{MM}} \otimes \rho_R) = \rho_{\overline{MM}} \otimes \rho_R \).
  - Since relative entropy can only decrease under the action of a CPTP map, \( S(\rho_{\overline{M}M\overline{R}} \parallel \rho_{\overline{MM}} \otimes \rho_R) = S(\text{Tr}_M \rho_{\overline{M}M\overline{R}} \parallel \text{Tr}_M \rho_{\overline{MM}} \otimes \rho_R) \).
  - Using \( S(\rho_{AB} \parallel \rho_A \otimes \rho_B) = S(A) + S(B) - S(AB) \) and the fact that \( \rho_{\overline{M}M\overline{R}} \) is pure, we get the desired result.
Information-theoretic condition for error correction

M is correctable iff $S(M\overline{M}) = S(M) - S(M)$ for any code state $\rho$.

- Obvious for pure states.
- Let $\rho_{M\overline{M}}$ be a code state and $\rho_{M\overline{M}R}$ its purification.
- By assumption, there exists $R$ on $\Lambda$ such that $R(\text{Tr}_M \rho_{M\overline{M}R}) = \rho_{M\overline{M}R}$ and $R(\text{Tr}_M \rho_{M\overline{M}} \otimes \rho_R) = \rho_{M\overline{M}} \otimes \rho_R$.
- Since relative entropy can only decrease under the action of a CPTP map, $S(\rho_{M\overline{M}R} \parallel \rho_{M\overline{M}} \otimes \rho_R) = S(\text{Tr}_M \rho_{M\overline{M}R} \parallel \text{Tr}_M \rho_{M\overline{M}} \otimes \rho_R)$
- Using $S(\rho_{AB} \parallel \rho_A \otimes \rho_B) = S(A) + S(B) - S(AB)$ and the fact that $\rho_{M\overline{M}R}$ is pure, we get the desired result.
Capacity-Stability Tradeoff

Tool 1

Information-theoretic condition for error correction

\( M \) is correctable iff \( S(\rho_{MM}) = S(\rho_M) - S(\rho) \) for any code state \( \rho \).

- Obvious for pure states.
- Let \( \rho_{MM} \) be a code state and \( \rho_{MMR} \) its purification.
- By assumption, there exists \( \mathcal{R} \) on \( \Lambda \) such that \( \mathcal{R}(\text{Tr}_M \rho_{MMR}) = \rho_{MRR} \) and \( \mathcal{R}(\text{Tr}_M \rho_{MM} \otimes \rho_R) = \rho_{MM} \otimes \rho_R \).
- Since relative entropy can only decrease under the action of a CPTP map, \( S(\rho_{MRR} \| \rho_{MM} \otimes \rho_R) = S(\text{Tr}_M \rho_{MRR} \| \text{Tr}_M \rho_{MM} \otimes \rho_R) \).
- Using \( S(\rho_{AB} \| \rho_A \otimes \rho_B) = S(A) + S(B) - S(AB) \) and the fact that \( \rho_{MRR} \) is pure, we get the desired result.
Information-theoretic condition for error correction

\[ M \text{ is correctable iff } S(M\bar{M}) = S(\overline{M}) - S(M) \text{ for any code state } \rho. \]

- Obvious for pure states.
- Let \( \rho_{M\bar{M}} \) be a code state and \( \rho_{M\bar{M}R} \) its purification.
- By assumption, there exists \( \mathcal{R} \) on \( \Lambda \) such that
  \[ \mathcal{R}(\text{Tr}_M \rho_{M\bar{M}R}) = \rho_{M\bar{M}R} \text{ and } \mathcal{R}(\text{Tr}_M \rho_{M\bar{M}} \otimes \rho_R) = \rho_{M\bar{M}} \otimes \rho_R. \]
- Since relative entropy can only decrease under the action of a CPTP map,
  \[ S(\rho_{M\bar{MR}} \| \rho_{M\bar{M}} \otimes \rho_R) = S(\text{Tr}_M \rho_{M\bar{MR}} \| \text{Tr}_M \rho_{M\bar{M}} \otimes \rho_R). \]
- Using \( S(\rho_{AB} \| \rho_A \otimes \rho_B) = S(A) + S(B) - S(AB) \) and the fact that \( \rho_{M\bar{M}R} \) is pure, we get the desired result.
### Tool 2

#### Union of correctable regions

Let $M_1$ and $M_2$ be correctable distant regions and suppose that $\partial M_1$ is also correctable. Then, $M_1 \cup M_2$ is correctable.

- Trivial for syndrome-based error correction (e.g. stabilizer codes).
- We will prove the Knill-Laflamme condition $\Pi_{O_{M_1} \otimes O_{M_2}} \Pi \propto \Pi$.
- The holographic disentangling lemma applied to $M_1$ implies that $\Pi = V_B V_C \eta_{AB} \langle \eta_{AB} | \otimes \nu_{B^2C_1} \langle \nu_{B^2C_1} | \otimes P_{C_2D} V_B^\dagger V_C^\dagger$.
- So $\Pi_{O_{M_1} \otimes O_{M_2}} \Pi = f(O_{M_1}) \Pi_{O_{M_2}} \Pi \propto \Pi$ where $f(O_{M_1}) = \langle \eta_{AB} | \langle \nu_{B^2C_1} | V_B^\dagger O_{M_1} V_B \nu_{B^2C_1} \eta_{AB} \rangle$. 
Tool 2

Union of correctable regions

Let $M_1$ and $M_2$ be correctable distant regions and suppose that $\partial M_1$ is also correctable. Then, $M_1 \cup M_2$ is correctable.

- Trivial for syndrome-based error correction (e.g. stabilizer codes).
- We will prove the Knill-Laflamme condition $\Pi O_{M_1} \otimes O_{M_2} \Pi \propto \Pi$.
- The holographic disentangling lemma applied to $M_1$ implies that $\Pi = V_B V_C |\eta_{AB}^1\rangle \langle \eta_{AB}^1| \otimes |\nu_{B^2 C^1}\rangle \langle \nu_{B^1 C^1}| \otimes P_{C^2 D} V_B^\dagger V_C^\dagger$.
- So $\Pi O_{M_1} \otimes O_{M_2} \Pi = f(O_{M_1}) \Pi O_{M_2} \Pi \propto \Pi$
- where $f(O_{M_1}) = \langle \eta_{AB}^1| \langle \nu_{B^2 C^1}| V_B^\dagger O_{M_1} V_B |\eta_{AB}^1\rangle |\nu_{B^2 C^1}\rangle$. 

David Poulin (Sherbrooke)
Let $M_1$ and $M_2$ be correctable distant regions and suppose that $\partial M_1$ is also correctable. Then, $M_1 \cup M_2$ is correctable.

- Trivial for syndrome-based error correction (e.g. stabilizer codes).
- We will prove the Knill-Laflamme condition $\Pi O_{M_1} \otimes O_{M_2} \Pi \propto \Pi$.
  
  The holographic disentangling lemma applied to $M_1$ implies that
  \[
  \Pi = V_B V_C |\eta_{AB}^1\rangle\langle\eta_{AB}^1| \otimes |\nu_{B^2C^1}\rangle\langle\nu_{B^1C^1}| \otimes P_{C^2D} V_B^\dagger V_C^\dagger.
  \]
  
  So $\Pi O_{M_1} \otimes O_{M_2} \Pi = f(O_{M_1}) \Pi O_{M_2} \Pi \propto \Pi$
  
  where $f(O_{M_1}) = \langle\eta_{AB}^1|\langle\nu_{B^2C^1}|V_B^\dagger O_{M_1} V_B^B|\eta_{AB}^1\rangle|\nu_{B^2C^1}\rangle$. 

\[\begin{array}{c}
\framebox{M_1} \\
A \quad B \\
C \quad D
\end{array}\]

\[\begin{array}{c}
\framebox{M_2}
\end{array}\]
Let $M_1$ and $M_2$ be correctable distant regions and suppose that $\partial M_1$ is also correctable. Then, $M_1 \cup M_2$ is correctable.

- Trivial for syndrome-based error correction (e.g. stabilizer codes).
- We will prove the Knill-Laflamme condition $\Pi O_{M_1} \otimes O_{M_2} \Pi \propto \Pi$.
- The holographic disentangling lemma applied to $M_1$ implies that $\Pi = V_B V_C |\eta_{AB}^1\rangle \langle \eta_{AB}^1| \otimes |\nu_{B^2C^1}\rangle \langle \nu_{B^1C^1}| \otimes P_{C^2D} V_B^\dagger V_C^\dagger$.
- So $\Pi O_{M_1} \otimes O_{M_2} \Pi = f(O_{M_1}) \Pi O_{M_2} \Pi \propto \Pi$ where $f(O_{M_1}) = \langle \eta_{AB}^1| \nu_{B^2C^1} |V_B^\dagger O_{M_1} V^B |\eta_{AB}^1\rangle |\nu_{B^2C^1}\rangle$.
Let $M_1$ and $M_2$ be correctable distant regions and suppose that $\partial M_1$ is also correctable. Then, $M_1 \cup M_2$ is correctable.

Trivial for syndrome-based error correction (e.g. stabilizer codes).

We will prove the Knill-Laflamme condition $\Pi O_{M_1} \otimes O_{M_2} \Pi \propto \Pi$.

The holographic disentangling lemma applied to $M_1$ implies that

$$\Pi = V_B V_C |\eta_{AB1}\rangle \langle \eta_{AB1}| \otimes |\nu_{B2C1}\rangle \langle \nu_{B1C1}| \otimes P_{C2D} V_B^\dagger V_C^\dagger.$$ 

So $\Pi O_{M_1} \otimes O_{M_2} \Pi = f(O_{M_1}) \Pi O_{M_2} \Pi \propto \Pi$ where

$$f(O_{M_1}) = \langle \eta_{AB1}| \langle \nu_{B2C1}| V_B^\dagger O_{M_1} V^B |\eta_{AB1}\rangle |\nu_{B2C1}\rangle.$$
Squares have perimeter \( \approx d \).

Region A (union of blue squares) is correctable.
Region B (union of red squares) is correctable.

Let's apply the information theoretic conditions to maximally mixed code state \( \rho = \Pi/\text{Tr}(\Pi) \) in two different ways:

Using \( S(BC) \leq S(B) + S(C) \) and \( S(AC) \leq S(A) + S(C) \):

\[
2S(ABC) \leq 2S(C)
\]

\( S(ABC) = k \).
\( S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2}. \)
Squares have perimeter $\approx d$.

Region $A$ (union of blue squares) is correctable.

Region $B$ (union of red squares) is correctable.

Let's apply the information theoretic conditions to maximally mixed code state $\rho = \Pi / \text{Tr}(\Pi)$ in two different ways:

Using $S(BC) \leq S(B) + S(C)$ and $S(AC) \leq S(A) + S(C)$

$$2S(ABC) \leq 2S(C)$$

- $S(ABC) = k$.
- $S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2}$.
Proof of Capacity-Stability tradeoff

- Squares have perimeter $\approx d$.
- Region $A$ (union of blue squares) is correctable.
- Region $B$ (union of red squares) is correctable.

Let's apply the information theoretic conditions to maximally mixed code state $\rho = \Pi/\text{Tr}(\Pi)$ in two different ways:

- Using $S(BC) \leq S(B) + S(C)$ and $S(AC) \leq S(A) + S(C)$
  \[ 2S(ABC) \leq 2S(C) \]
  \[ S(ABC) = k. \]
  \[ S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2}. \]
Squares have perimeter \( \approx d \).
Region \( A \) (union of blue squares) is correctable.
Region \( B \) (union of red squares) is correctable.

Let's apply the information theoretic conditions to maximally mixed code state \( \rho = \Pi / \text{Tr}(\Pi) \) in two different ways:

\[
S(ABC) = S(BC) - S(A) \quad \text{and} \quad S(ABC) = S(AC) - S(B)
\]

Using \( S(BC) \leq S(B) + S(C) \) and \( S(AC) \leq S(A) + S(C) \):

\[
2S(ABC) \leq 2S(C)
\]

- \( S(ABC) = k \).
- \( S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2} \).
Squares have perimeter \( \approx d \).
Region A (union of blue squares) is correctable.
Region B (union of red squares) is correctable.
Let's apply the information theoretic conditions to maximally mixed code state \( \rho = \Pi / \text{Tr}(\Pi) \) in two different ways:

\[
S(ABC) = S(BC) - S(A) \quad \text{and} \quad S(ABC) = S(AC) - S(B)
\]

Using \( S(BC) \leq S(B) + S(C) \) and \( S(AC) \leq S(A) + S(C) \)

\[
2S(ABC) \leq 2S(C)
\]

- \( S(ABC) = k \).
- \( S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2} \).
Proof of Capacity-Stability tradeoff

- Squares have perimeter $\approx d$.
- Region $A$ (union of blue squares) is correctable.
- Region $B$ (union of red squares) is correctable.
- Let’s apply the information theoretic conditions to maximally mixed code state $\rho = \Pi / \text{Tr} (\Pi)$ in two different ways:

\[ S(ABC) \leq S(B) + S(C) - S(A) \quad \text{and} \quad S(ABC) \leq S(A) + S(C) - S(B) \]

Using $S(BC) \leq S(B) + S(C)$ and $S(AC) \leq S(A) + S(C)$

\[ 2S(ABC) \leq 2S(C) \]

- $S(ABC) = k$.
- $S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2}$. 
Proof of Capacity-Stability tradeoff

- Squares have perimeter $\approx d$.
- Region $A$ (union of blue squares) is correctable.
- Region $B$ (union of red squares) is correctable.
- Let's apply the information theoretic conditions to maximally mixed code state $\rho = \Pi / \text{Tr}(\Pi)$ in two different ways:

$$S(ABC) \leq S(B) + S(C) - S(A) \quad \text{and} \quad S(ABC) \leq S(A) + S(C) - S(B)$$

Using $S(BC) \leq S(B) + S(C)$ and $S(AC) \leq S(A) + S(C)$

$$2S(ABC) \leq 2S(C)$$

- $S(ABC) = k$.
- $S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2}$. 

David Poulin (Sherbrooke)

2D Quantum Memories

Coogee’13 19 / 25
Proof of Capacity-Stability tradeoff

- Squares have perimeter \( \approx d \).
- Region A (union of blue squares) is correctable.
- Region B (union of red squares) is correctable.
- Let's apply the information theoretic conditions to maximally mixed code state \( \rho = \Pi/\text{Tr}(\Pi) \) in two different ways:

\[
S(ABC) \leq S(B) + S(C) - S(A) \quad \text{and} \quad S(ABC) \leq S(A) + S(C) - S(B)
\]

Using \( S(BC) \leq S(B) + S(C) \) and \( S(AC) \leq S(A) + S(C) \)

\[
2S(ABC) \leq 2S(C)
\]

- \( S(ABC) = k \).
- \( S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2} \).
Squares have perimeter $\approx d$.
Region $A$ (union of blue squares) is correctable.
Region $B$ (union of red squares) is correctable.

Let's apply the information theoretic conditions to maximally mixed code state $\rho = \Pi / \text{Tr}(\Pi)$ in two different ways:

\[
S(ABC) \leq S(B) + S(C) - S(A) \quad \text{and} \quad S(ABC) \leq S(A) + S(C) - S(B)
\]

Using $S(BC) \leq S(B) + S(C)$ and $S(AC) \leq S(A) + S(C)$,

\[
2S(ABC) \leq 2S(C)
\]

- $S(ABC) = k$.
- $S(C) \leq |C| \propto \text{number of circles} \propto \frac{n}{d^2}$. 
Outline

1. 2D Commuting Projector Codes
2. Holographic Disentangling Lemma
3. Holographic Minimum Distance
4. Capacity-Stability Tradeoff
5. String-Like Logical Operators
6. Open Questions
String-like logical operators

There exists a non-trivial logical operator supported on a string-like region.

- Well known for Kitaev's toric code.
- Intuitive for known models that support anyons:
  - The ground state can be changed by dragging an anyon around a topologically non-trivial loop.
  - This process is realized on a string, and generated a logical operation.
- Relation to thermal instability?
String-like logical operators

There exists a non-trivial logical operator supported on a string-like region.

- Well known for Kitaev’s toric code.
- Intuitive for known models that support anyons:
  - The ground state can be changed by dragging an anyon around a topologically non-trivial loop.
  - This process is realized on a string, and generated a logical operation.
- Relation to thermal instability?
String-like logical operators

There exists a non-trivial logical operator supported on a string-like region.

- Well known for Kitaev’s toric code.
- Intuitive for known models that support anyons:
  - The ground state can be changed by dragging an anyon around a topologically non-trivial loop.
  - This process is realized on a string, and generated a logical operation.
- Relation to thermal instability?
String-like logical operators

There exists a non-trivial logical operator supported on a string-like region.

- Well known for Kitaev’s toric code.
- Intuitive for known models that support anyons:
  - The ground state can be changed by dragging an anyon around a topologically non-trivial loop.
  - This process is realized on a string, and generated a logical operation.
- Relation to thermal instability?
String-like logical operators

There exists a non-trivial logical operator supported on a string-like region.

- Well known for Kitaev’s toric code.
- Intuitive for known models that support anyons:
  - The ground state can be changed by dragging an anyon around a topologically non-trivial loop.
  - This process is realized on a string, and generated a logical operation.

- Relation to thermal instability?
String-like logical operators

There exists a non-trivial logical operator supported on a string-like region.

- Well known for Kitaev’s toric code.
- Intuitive for known models that support anyons:
  - The ground state can be changed by dragging an anyon around a topologically non-trivial loop.
  - This process is realized on a string, and generated a logical operation.
- Relation to thermal instability?
Proof, part 1

There exists a string-like region that is not-correctable.

- Let $M$ be a string-like region.
- Suppose $M$ is correctable.
- Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
- If either $\partial M_L$ or $\partial M_R$ are not correctable, we are done.
- Otherwise $\partial M = \partial M_L \cup \partial M_R$ is correctable, and therefore $M \cup \partial M$ is correctable.
- Continue until we arrive at $\Lambda$ is correctable, which is impossible.
There exists a string-like region that is not-correctable.

- Let $M$ be a string-like region.
- Suppose $M$ is correctable.
- Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
- If either $\partial M_L$ or $\partial M_R$ are not correctable, we are done.
- Otherwise $\partial M = \partial M_L \cup \partial M_R$ is correctable, and therefore $M \cup \partial M$ is correctable.
- Continue until we arrive at $\Lambda$ is correctable, which is impossible.
Proof, part 1

There exists a string-like region that is not-correctable.

- Let $M$ be a string-like region.
- Suppose $M$ is correctable.
  - Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
  - If either $\partial M_L$ or $\partial M_R$ are not correctable, we are done.
  - Otherwise $\partial M = \partial M_L \cup \partial M_R$ is correctable, and therefore $M \cup \partial M$ is correctable.
  - Continue until we arrive at $\Lambda$ is correctable, which is impossible.
Proof, part 1

There exists a string-like region that is not-correctable.

Let $M$ be a string-like region.
Suppose $M$ is correctable.
Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
If either $\partial M_L$ or $\partial M_R$ are not correctable, we are done.
Otherwise $\partial M = \partial M_L \cup \partial M_R$ is correctable, and therefore $M \cup \partial M$ is correctable.
Continue until we arrive at $\Lambda$ is correctable, which is impossible.
There exists a string-like region that is not-correctable.

- Let $M$ be a string-like region.
- Suppose $M$ is correctable.
- Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
- If either $\partial M_L$ or $\partial M_R$ are not correctable, we are done.
- Otherwise $\partial M = \partial M_L \cup \partial M_R$ is correctable, and therefore $M \cup \partial M$ is correctable.
- Continue until we arrive at $\Lambda$ is correctable, which is impossible.
There exists a string-like region that is not-correctable.

- Let $M$ be a string-like region.
- Suppose $M$ is correctable.
- Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
- If either $\partial M_L$ or $\partial M_R$ are not correctable, we are done.
- Otherwise $\partial M = \partial M_L \cup \partial M_R$ is correctable, and therefore $M \cup \partial M$ is correctable.
- Continue until we arrive at $\Lambda$ is correctable, which is impossible.
Proof, part 1

There exists a string-like region that is not-correctable.

- Let $M$ be a string-like region.
- Suppose $M$ is correctable.
- Consider its boundary $\partial M = \partial M_L \cup \partial M_R$.
- If either $\partial M_L$ or $\partial M_R$ are not correctable, we are done.
- Otherwise $\partial M = \partial M_L \cup \partial M_R$ is correctable, and therefore $M \cup \partial M$ is correctable.
- Continue until we arrive at $\Lambda$ is correctable, which is impossible.
Let $M$ be a non-correctable string-like region.

- There exists $O_M$ such that $\Pi O_M \Pi \not\propto \Pi$.
- Let $\Pi_M = \prod_{X \cap M \neq \emptyset} P_X$
- Then $X = \Pi_M O_M \Pi_M$ is a non-trivial logical operator supported on $M \cup \partial M$.
- Any function of $X$, e.g. $\exp(-iX\theta)$, is also a logical operator with the same support.
Let $M$ be a non-correctable string-like region.

There exists $O_M$ such that $\Pi O_M \Pi \nleq \Pi$.

Let $\Pi_M = \prod_{X \cap M \neq \emptyset} P_X$

Then $X = \Pi_M O_M \Pi_M$ is a non-trivial logical operator supported on $M \cup \partial M$.

Any function of $X$, e.g. $\exp(-iX\theta)$, is also a logical operator with the same support.
Let $M$ be a non-correctable string-like region. There exists $O_M$ such that $\Pi O_M \Pi \propto \Pi$. Let $\Pi_M = \prod_{X \cap M \neq \emptyset} P_X$. Then $X = \Pi_M O_M \Pi_M$ is a non-trivial logical operator supported on $M \cup \partial M$. Any function of $X$, e.g. $\exp(-iX\theta)$, is also a logical operator with the same support.
Proof, part 2

- Let $M$ be a non-correctable string-like region.
- There exists $O_M$ such that $\Pi O_M \Pi \propto \Pi$.
- Let $\Pi_M = \prod_{X \cap M \neq \emptyset} P_X$
- Then $X = \Pi_M O_M \Pi_M$ is a non-trivial logical operator supported on $M \cup \partial M$.
- Any function of $X$, e.g. $\exp(-iX\theta)$, is also a logical operator with the same support.
Proof, part 2

- Let $M$ be a non-correctable string-like region.
- There exists $O_M$ such that $\Pi O_M \Pi \propto \Pi$.
- Let $\Pi_M = \prod_{X \cap M \neq \emptyset} P_X$
- Then $X = \Pi_M O_M \Pi_M$ is a non-trivial logical operator supported on $M \cup \partial M$.
- Any function of $X$, e.g. $\exp(-iX\theta)$, is also a logical operator with the same support.
Outline

1. 2D Commuting Projector Codes
2. Holographic Disentangling Lemma
3. Holographic Minimum Distance
4. Capacity-Stability Tradeoff
5. String-Like Logical Operators
6. Open Questions
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{\frac{2}{D-1}}$

- How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

- String-like logical operators $\Rightarrow$ constant energy barrier.
  - This is not directly related to thermal instability.
  - 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
  - What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
  - Can we characterize all string-like logical operators?
  - Relation between commuting projector codes and anyon models.

- Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).
  - Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

- Extension to subsystem codes?
  - With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{D-1}$

How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

String-like logical operators $\Rightarrow$ constant energy barrier.
- This is not directly related to thermal instability.
- 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
- What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
- Can we characterize all string-like logical operators?
- Relation between commuting projector codes and anyon models.

Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).
- Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

Extension to subsystem codes?
- With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{\frac{2}{D-1}}$

How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

String-like logical operators $\Rightarrow$ constant energy barrier.

- This is not directly related to thermal instability.
- 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
- What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
- Can we characterize all string-like logical operators?
- Relation between commuting projector codes and anyon models.

Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).

- Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

Extension to subsystem codes?

- With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{D-1}$

How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

String-like logical operators $\Rightarrow$ constant energy barrier.

This is not directly related to thermal instability.

2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.

What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).

Can we characterize all string-like logical operators?

Relation between commuting projector codes and anyon models.

Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).

Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

Extension to subsystem codes?

With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{D-2}$

- How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

**String-like logical operators $\Rightarrow$ constant energy barrier.**

- This is not directly related to thermal instability.
- 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
- What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).

- Can we characterize all string-like logical operators?
- Relation between commuting projector codes and anyon models.

- Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).
  - Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

- Extension to subsystem codes?
  - With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{D-1}$

- How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

- String-like logical operators $\Rightarrow$ constant energy barrier.
  - This is not directly related to thermal instability.
  - 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
  - What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).

- Can we characterize all string-like logical operators?
- Relation between commuting projector codes and anyon models.

- Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).
  - Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

- Extension to subsystem codes?
  - With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/\sqrt{D-1}$

- How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

- String-like logical operators $\Rightarrow$ constant energy barrier.
  - This is not directly related to thermal instability.
  - 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
  - What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
  - Can we characterize all string-like logical operators?
    - Relation between commuting projector codes and anyon models.

- Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).
  - Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

- Extension to subsystem codes?
  - With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{D-1}$

- How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

String-like logical operators $\Rightarrow$ constant energy barrier.

- This is not directly related to thermal instability.
- 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
- What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
- Can we characterize all string-like logical operators?
- Relation between commuting projector codes and anyon models.

- Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).
  - Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.
- Extension to subsystem codes?
  - With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq \frac{cn}{d^{D-1}}$

- How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

**String-like logical operators $\Rightarrow$ constant energy barrier.**
- This is not directly related to thermal instability.
- 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
- What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
- Can we characterize all string-like logical operators?
- Relation between commuting projector codes and anyon models.

**Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).**
- Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

**Extension to subsystem codes?**
- With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{D-1}$

- How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

String-like logical operators $\Rightarrow$ constant energy barrier.

- This is not directly related to thermal instability.
- 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
- What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).
- Can we characterize all string-like logical operators?
- Relation between commuting projector codes and anyon models.

Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).

- Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

Extension to subsystem codes?

- With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{2/D-1}$

How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

String-like logical operators $\Rightarrow$ constant energy barrier.

This is not directly related to thermal instability.

2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.

What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).

Can we characterize all string-like logical operators?

Relation between commuting projector codes and anyon models.

Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).

Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

Extension to subsystem codes?

With local stabilizer (Bombin) and without (Bacon-Shor).
All our results extend to $D$-dimensional lattices, e.g. $k \leq cn/d^{D-1}$

- How about infinite dimensions (LDPC codes)? (Delfosse & Zémor)

String-like logical operators $\Rightarrow$ constant energy barrier.

- This is not directly related to thermal instability.
- 2D Ising model has an energy barrier $\propto \sqrt{n}$, but an energy $\propto n$ at finite temperature.
- What matters is entropy (for a given energy, many more configurations many with small error droplets than with a large one).

Can we characterize all string-like logical operators?

Relation between commuting projector codes and anyon models.

Extend to frustration-free Hamiltonians (and therefore to all gapped Hamiltonians, i.e. Hastings).

- Use proof techniques to show that gapped Hamiltonian $\in$ QCMA.

Extension to subsystem codes?

- With local stabilizer (Bombin) and without (Bacon-Shor).