Exploring Quantum Chaos with Quantum Information Processors

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Simulating quantum systems: the readout problem.

The QDC1 model of computation.

The elementary scattering circuit.

Looking for symmetries in complex quantum systems.
  - Random matrix conjecture.
  - Estimating the form factors.

Hypersensitivity to perturbations: fidelity decay.
  - DQC1 algorithm.
  - Experimental results.

Decoherence and dynamical instability.

Entanglement, decoherence and quantum-computational speed-up.

Conclusion.
Readout problem

In a physical experiment, a system evolves from its initial state and is finally measured in some basis.
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We know that quantum computer can be use to simulate the evolution of some quantum systems (Zalka, Lloyd, ...).

\[ U(t) \approx U_n \ldots U_2 U_1 \]

Universal set
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The ability to efficiently simulate the dynamics \( U(t) \) alone can be used to extract interesting physical quantities.
Liquid state NMR QIP

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\[ H \simeq \sum_i \vec{\mu}_i \cdot \vec{B} + \sum_{i \neq j} J_{ij} \vec{\mu}_i \vec{\mu}_j, \quad \omega \gg J \quad \text{and} \quad \beta \ll 1, \quad \text{so} \]

\[ \rho \simeq \frac{1}{Z} \left( 1 - \beta \sum_i \omega_i \sigma_z \right) \]
The initial state is thermal $\rho = \frac{1}{Z} e^{-\beta H}$.

$H \simeq \sum_i \vec{\mu}_i \cdot \vec{B} + \sum_{i \neq j} J_{ij} \vec{\mu}_i \vec{\mu}_j$, $\omega \gg J$ and $\beta \ll 1$, so

$$\rho \simeq \frac{1}{Z} \left( 1 - \beta \sum_i \omega_i \sigma_{zi} \right)$$

Apply an inhomogeneous magnetic field to induce a random phase:

$$\rho \rightarrow \frac{1}{Z} \left( 1 + \frac{\beta \omega n}{2^n} |0\ldots00\rangle\langle 0\ldots00| \right)$$
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Pseudo pure state, the identity part does not contribute to the computation.

This exponential decay can be avoided using \textit{algorithmic cooling}, but the overhead is $\sim 10^{10}$.
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Add restrictions to the “standard” model of quantum computation

1. Only a single pseudo-pure qubit.

$$\rho = \left( \frac{1 - \epsilon}{2} \mathbb{1} + \epsilon |0\rangle \langle 0| \right) \otimes \frac{1}{2^n} \mathbb{1}$$
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Can we do something non trivial with this?

E. Knill and R. Laflamme, 1998
Remarks:

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From a computational complexity point of view:

$|00...0\rangle \langle 00...0| \otimes 1^2^n$
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Caltech, November 2004 – p.6
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4. One pure qubit $\Leftrightarrow \log(n)$ pure qubits: cost $\text{poly}(n)$.
   - From a computational complexity point of view:

$$\underbrace{|00\ldots0\rangle\langle 00\ldots0|}_{\ln n} \otimes \frac{1}{2^n} \mathbb{1}$$
\[ \gamma = Tr\{U \rho\} \]
If $U$ can be implemented efficiently and $\rho$ prepared efficiently, we can evaluate $\text{Tr}\{U \rho\}$ within accuracy $\delta$ in time $\text{poly}(n)/\delta^2$.

\[
\rho' = \frac{1}{2} \begin{pmatrix}
1 + \text{Re}\{\gamma\} & i\text{Im}\{\gamma\} \\
-i\text{Im}\{\gamma\} & 1 - \text{Re}\{\gamma\}
\end{pmatrix}
\]
Scattering circuit

\[
|0\rangle\langle 0| \xrightarrow{H} \langle \sigma_k \rangle \\
\rho \xrightarrow{U} \gamma = Tr\{U \rho\}
\]

\[
\rho_1' = \frac{1}{2} \begin{pmatrix}
1 + Re\{\gamma\} & iIm\{\gamma\} \\
-iIm\{\gamma\} & 1 - Re\{\gamma\}
\end{pmatrix}
\]

\[
k = z : \langle \sigma_z \rangle = Tr\{\rho_1' \sigma_z\} = Re\{\gamma\} = Re[Tr\{\rho U\}]
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k = y : \langle \sigma_y \rangle = Tr\{\rho_1' \sigma_y\} = Im\{\gamma\} = Im[Tr\{\rho U\}]
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If \( U \) can be implemented efficiently and \( \rho \) prepared efficiently, we can evaluate \( Tr\{\rho U\} \) within accuracy \( \delta \) in time \( poly(n)/\delta^2 \).
We can evaluate $Tr\{\rho U\}$.
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Control $U$ and learn about $\rho$: quantum state tomography.
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These algorithms require an exponential repetition of the scattering circuit.

- Can we get useful partial information about $U$?
- DQC1 imposes restrictions on $\rho$. We will focus on $\rho \propto 1$.

$$\gamma = \frac{1}{N} \text{Tr}\{U\} = \frac{1}{N} \sum_j \lambda_j$$
Doing the obvious thing

\[ \frac{1}{N} Tr\{U\} = \frac{1}{N} \sum_j e^{i\phi_j} \]

\[ = \frac{1}{N} \sum_j \vec{v}_j \]

Is there an intuitive way of understanding these two different behaviors? Can we learn something useful about the dynamics of the system with this simple algorithm?

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Quantum control

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Does my system possess any symmetries?
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- Quantum information processing is the art of manipulating complex quantum systems.
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- *Does my system possess any symmetries?*

Systems with as many symmetries as degrees of freedom are integrable.

- These are good candidates for quantum information processors.
Quantum control

Quantum information processing is the art of manipulating complex quantum systems.

Control a quantum system = exploit its symmetries (QECC, NSS, bang-bang)

Does my system possess any symmetries?

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These are good candidates for quantum information processors.

Chaotic systems possess no or very few symmetries.

They are hard to control.

They are dynamically unstable (sensitive to perturbations).
Looking for symmetries

Resonances of Thorium 232
Looking for symmetries

List of resonances

<table>
<thead>
<tr>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$E_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>1.4</td>
<td>1.6</td>
<td>1.8</td>
</tr>
<tr>
<td>2.0</td>
<td>2.2</td>
<td>2.4</td>
<td>2.6</td>
</tr>
<tr>
<td>3.0</td>
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<td>4.0</td>
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Table 1: Resonance levels (parameters of $1^p$)

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Level spacing

**Fig. 10.** Histogram of the observed distribution of nearest-neighbor level spacings $s = D/D_0$ for Th. The three theoretical curves correspond respectively to random, orthogonal ($\beta = 1$) and unitary ($\beta = 2$). The orthogonal ($\beta = 1$) curve is the favored theoretical distribution.
Looking for symmetries

Wigner’s intuition: The main characteristics of the spectrum of can be reproduced by random matrices which possess the same symmetries as the system.
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\[ f(U) \simeq \int f(V) P(V) dV. \]

- \( P(V) \neq 0 \) only for those \( V \) with the same symmetries as \( U \).
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Symmetries = Block diagonal

\[ U = \exp \left\{ -\frac{iHt}{\hbar} \right\} = \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_d \end{pmatrix} \]

Mixture of i.i.d variables → Poisson distribution
Looking for symmetries

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Requires accuracy \( \frac{1}{\sqrt{N}} \) so overhead is \( N \): square-root improvement over brute force classical computation.

Fidelity decay

Quantum map $U$, e.g. $U = \exp\{-iHt\}$. How sensitive is $U$ to perturbation?
Fidelity decay

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Perturbed quantum map $U_p$, e.g.

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U_p = U \exp\{-i\delta V\}^K
\]

Fidelity after \( n \) steps (Lodschmidt echo):

\[
F_n(\psi) = \left| \left\langle \psi \right| \left( U^n \right)^\dagger U^n_p |\psi\rangle \right|^2
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Fidelity decay

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Fidelity after $n$ steps (Lodschmidt echo):

$$F_n(\psi) = |\langle \psi | (U^n)^\dagger U_p^n | \psi \rangle|^2$$

State independent signature (average over $\psi$):

$$\overline{F_n} = \int F_n(\psi) d\psi = \frac{|Tr\{(U^n)^\dagger U_p^n\}|^2}{N^2 + N} + N$$
Fidelity decay

Peres’ conjecture: $F_n(\psi)$ is a signature of quantum chaos (dynamical instability).

- Chaotic systems have an exponential decay $\exp(-\Gamma n)$.
- Regular systems have a polynomial decay $1/poly(n)$. 
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Fidelity decay measures the relative randomness of $U$ and $K$.

- Universal response to perturbation.
- “Simple” perturbation produce good signature of quantum chaos.
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- Fidelity decay measures the relative randomness of $U$ and $K$.
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J. Emerson, Y.S. Weinstein, S. Lloyd, and D.G. Cory, 2002
Trace circuit


\[ |0\rangle \langle 0 | \quad H \quad U \quad H \quad \langle \sigma_k \rangle \]

\[
\frac{1}{2^n} \quad U
\]

\[
k = y : \langle \sigma_y \rangle = \frac{1}{N} \text{Im} [\text{Tr}\{U\}] \\
k = z : \langle \sigma_z \rangle = \frac{1}{N} \text{Re} [\text{Tr}\{U\}] \\
F_n = \frac{|\text{Tr}\{(U^n)^\dagger U_p^n\}|^2 + N}{N^2 + N}, \quad U_p = UK
\]
Trace circuit


\[ |0\rangle\langle 0| \quad \overset{\text{H}}{\longrightarrow} \quad \frac{1}{2^n} \overset{U}{\longrightarrow} \quad \langle \sigma_k \rangle \]

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\[ k = z : \langle \sigma_z \rangle = \frac{1}{N} \text{Re} \left[ \text{Tr} \{ U \} \right] \]

\[ F_n = \frac{\left| \text{Tr} \{ (U^n)^\dagger U_p^n \} \right|^2 + N}{N^2 + N}, \quad U_p = UK \]

All we have to do it replace \( U \) in the above circuit by \( (U^n)^\dagger (UK)^n \).
Since the $U$'s and $U^\dagger$'s annihilate each other when the $K$'s are absent.

Measure $F_n$ within accuracy $\epsilon$ with error probability $p$ in time $nO((\log(1/p))^2)$.
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Quantum circuit

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Since the $U^\dagger$ are made after the final measurement.

Measure $\overline{F_n}$ within accuracy $\epsilon$ with error probability $p$ in time

$$nO\left(\frac{\log(1/p)}{\epsilon^2}\right)T$$
Experimental implementation

Colm Ryan, et al. (IQC)

[Graphs and images of chemical structures]
We use the circuit to model a system interacting with an environment, the system is initially in a pure state $\alpha|0\rangle + \beta|1\rangle$: 

$$\rho_0 \xrightarrow{\frac{1}{N}} \begin{array}{c}
\rho_n \\
\begin{array}{cccc}
U & K & U & K \\
\vdots & \ddots & \ddots & \ddots \\
U & K
\end{array}
\end{array}$$

The decoherence rate is governed by the average fidelity decay rate of the environment! The decoherence rate depends on the strength of the coupling to the environment but also on the dynamics of the environment.
Decoherence

We use the circuit to model a system interacting with an environment, the system is initially in a pure state $\alpha|0\rangle + \beta|1\rangle$:

$$\rho_0 = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \Rightarrow \rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\gamma(n) \\ \alpha^*\beta\gamma(n)^* & |\beta|^2 \end{pmatrix}$$
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$$|\gamma(n)| = \left\{ \frac{F_n}{N} \right\}^{1/2} : \text{The decoherence rate is governed by the average fidelity decay rate of the environment!}$$
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$|\gamma(n)| = \left\{ \frac{F_n}{N} \right\}^{1/2}$: The decoherence rate is governed by the average fidelity decay rate of the environment!

The decoherence rate depends on the strength of the coupling to the environment but also on the dynamics of the environment.
Using the "phase kick-back", we see that the final state is
\[ \rho_k = \frac{1}{N} \sum_j |\alpha_j\rangle \langle \alpha_j| \otimes |\phi_j\rangle \langle \phi_j| \]
where
\[ |\alpha_j\rangle = \alpha |0\rangle + \beta e^{i\theta_j} |1\rangle \]
and the \[ e^{i\theta_j} \] are the eigenvalues of
\[ S = U_k P_k \cdots U_2 P_2 U_1 P_1 U_1^* \cdots U_k^* . \]
This circuit does not produce any entanglement between the probe qubit and the rest, throughout the computation.
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\[ \rho_k = \frac{1}{N} \sum_j |\alpha_j\rangle\langle\alpha_j| \otimes |\phi_j\rangle\langle\phi_j| \]

where \( |\alpha_j\rangle = \alpha|0\rangle + \beta e^{i\theta_j} |1\rangle \) and the \( e^{i\theta_j} \) are the eigenvalues of \( S = U_k P_k \ldots U_2 P_2 U_1 P_1 U_1^\dagger U_2^\dagger \ldots U_k^\dagger \).
Using the "phase kick-back", we see that the final state is

\[ \rho_k = \frac{1}{N} \sum_j |\alpha_j\rangle\langle\alpha_j| \otimes |\phi_j\rangle\langle\phi_j| \]

where \( |\alpha_j\rangle = \alpha |0\rangle + \beta e^{is_j} |1\rangle \) and the \( e^{is_j} \) are the eigenvalues of \( S = U_k P_k \ldots U_2 P_2 U_1 P_1 U_1^\dagger U_2^\dagger \ldots U_k^\dagger \).

This circuit does not produce any entanglement between the probe qubit and the rest, throughout the computation.
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If no, is there a classical dynamics which can provide the sequence of "classical" states?
Conclusion

- DQC1 is interesting.
  - Measure the form factors to learn about symmetries.
  - Measure fidelity decay to learn about dynamical stability.
- Experimental benchmarking of quantum information processing.
- Decoherence rate is influenced by the dynamical properties of the environment.
  - Decoherence does not require entanglement.
- Entanglement is not the key ingredient to quantum computational speed-up.
- We can learn about fundamental physics by working on quantum computation, and vice versa.
References

