Exploring Quantum Chaos with Quantum Computers

Part II: Measuring signatures of quantum chaos on a quantum computer

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Outline

- Classical chaos
- The quest for quantum signatures
- Readout problem
- DQC1
- Random matrices & form factors
- Fidelity decay
  - Quantum algorithm
  - DQC1 algorithm
  - Decoherence and quantum chaos*
  - Absence of entanglement*

*Time permitting
Classical chaos

- Lyapunov exponents.
Classical chaos

Lyapunov exponents.

\[ dx(t) \approx dx e^{\lambda t} \]
Classical chaos

- Lyapunov exponents.

\[ dx(t) \approx dx e^{\lambda t} \]

- Kolmogorov-Sinai entropy, mixing of phase space.
The quest for quantum signatures

Problems with the classical signatures:

No notion of point in phase space, rather we have vectors in Hilbert space.

Positivity of distributions over phase space is not preserved.

Unitarity preserves distances between states:

\[ h(t) \] 

\[ i \]
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- Unitarity preserves distances between states:

\[ \langle \psi(t) | \phi(t) \rangle = \langle \psi(0) | \phi(0) \rangle \]
Readout problem

It has been known for some time that a quantum computer could be used to simulate the evolution of quantum systems (Zalka, Lloyd, ...).

\[ U(t) = U_n \ldots U_2 U_1 \]

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Then what?
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\text{Universal set}
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Then what?

**Classical simulation:**

\[
\psi = (0.2677 + 0.002i, \ldots 0.00098538 - 0.1i)^T.
\]
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Then what?

- Classical simulation:
  \[
  \psi = (0.2677 + 0.002i, \ldots 0.00098538 - 0.1i)^T.
  \]
- Quantum lab: single or multiple (macroscopically many) systems described by \( \psi \).
- Quantum computer: Ability to simulate evolution.
Readout problem

Do we need to prepare a special initial state?

Low temperature transport properties

$$\langle \psi_0 | e^{-i(H_0+V)} | \psi_k \rangle.$$
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Do we need to accumulate statistics? How many?

- Spectrum of a molecule.
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  - Spectrum of a molecule.

- etc.

The ability to efficiently simulate the dynamics \( U(t) \) alone can be used to extract interesting physical quantities.
Liquid state NMR QIP

Thermal state \( \rho = \frac{1}{Z} e^{-\beta H} \).
Liquid state NMR QIP

- Thermal state $\rho = \frac{1}{Z} e^{-\beta H}$.

- $H \approx \sum_i \vec{\mu}_i \cdot \vec{B} + \sum_{i \neq j} J_{ij} \vec{\mu}_i \vec{\mu}_j$ and $\beta \ll 1$, so

$$
\rho \approx \frac{1}{Z} \left( 1 - \beta \sum_i \omega_i \sigma_{zi} \right)
$$
Liquid state NMR QIP

- Thermal state $\rho = \frac{1}{Z} e^{-\beta H}$.

$$H \approx \sum_i \vec{\mu}_i \cdot \vec{B} + \sum_{i \neq j} J_{ij} \vec{\mu}_i \vec{\mu}_j \quad \text{and} \quad \beta \ll 1, \text{ so}$$

$$\rho \approx \frac{1}{Z} \left( \mathbf{1} - \beta \sum_i \omega_i \sigma_{zi} \right)$$

- Using a gradient field,

$$\rho \rightarrow \frac{1}{Z} \left( \mathbf{1} + \frac{\beta \omega n}{2^n} |0 \ldots 00\rangle \langle 0 \ldots 00| \right)$$

This exponential decay can be avoided using \textit{algorithmic cooling}, but the overhead is \( \approx 10^{10} \).
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Add restrictions to the “standard” model of quantum computation
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Add restrictions to the “standard” model of quantum computation

1. Only a single pseudo-pure qubit.

\[
\rho = \left( \frac{1 - \epsilon}{2} \mathbb{1} + \epsilon |0\rangle \langle 0| \right) \otimes \frac{1}{2^n} \mathbb{1}
\]
This exponential decay can be avoided using *algorithmic cooling*, but the overhead is $\simeq 10^{10}$.

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2. No projective measurement, rather $\langle \sigma_z \rangle$ within finite accuracy $\mu$. 
This exponential decay can be avoided using *algorithmic cooling*, but the overhead is $\sim 10^{10}$.

Add restrictions to the “standard” model of quantum computation

1. Only a single pseudo-pure qubit.

$$\rho = \left( \frac{1 - \epsilon}{2} \text{11} + \epsilon |0\rangle\langle 0| \right) \otimes \frac{1}{2^n} \text{11}$$

2. No projective measurement, rather $\langle \sigma_z \rangle$ within finite accuracy $\mu$.

Can we do something non trivial with this?
E. Knill and R. Laflamme, 1998

Remarks:

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4. One pure qubit $\Leftrightarrow \log(n)$ pure qubits: cost $\text{poly}(n)$.
Trace circuit

\[ |0\rangle \langle 0| \quad \text{H} \quad \text{H} \quad \langle \sigma_k \rangle \]

\[ \rho \quad \text{U} \]

Trace circuit

\[ |0\rangle\langle 0| \quad \xrightarrow{H} \quad \rho \quad \xrightarrow{U} \quad |\sigma_k\rangle \]


\[
\rho_0 = |0\rangle\langle 0| \otimes \rho
\]

\[
\rightarrow \frac{1}{2} (|0\rangle\langle 0| \otimes \rho + |0\rangle\langle 1| \otimes \rho + |1\rangle\langle 0| \otimes \rho + |1\rangle\langle 1| \otimes \rho)
\]

\[
\rightarrow \frac{1}{2} (|0\rangle\langle 0| \otimes \rho + |0\rangle\langle 1| \otimes \rho U^\dagger + |1\rangle\langle 0| \otimes U \rho + |1\rangle\langle 1| \otimes U \rho U^\dagger)
\]

\[
= \rho_f
\]
Trace circuit

\[ |0\rangle\langle 0| \quad \begin{array}{c} \text{H} \\ \text{U} \\ \text{H} \end{array} \quad \langle \sigma_k \rangle \]

\[ \rho \]


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\rho_0 = |0\rangle\langle 0| \otimes \rho \\
\rightarrow \frac{1}{2} (|0\rangle\langle 0| \otimes \rho + |0\rangle\langle 1| \otimes \rho + |1\rangle\langle 0| \otimes \rho + |1\rangle\langle 1| \otimes \rho) \\
\rightarrow \frac{1}{2} (|0\rangle\langle 0| \otimes \rho + |0\rangle\langle 1| \otimes \rho \rho U^\dagger + |1\rangle\langle 0| \otimes U \rho + |1\rangle\langle 1| \otimes U \rho U^\dagger) \\
= \rho_f
\]

\[
\rho_1 = Tr_n \{ \rho_f \} = \frac{1}{2} (|0\rangle\langle 0| + |0\rangle\langle 1| Tr \{ \rho U^\dagger \} + |1\rangle\langle 0| Tr \{ U \rho \} + |1\rangle\langle 1|)
\]
Trace circuit

With $\gamma = Tr\{U \rho\}$,
Trace circuit

\[ |0\rangle\langle 0| \quad \text{H} \quad \text{H} \quad \langle \sigma_k \rangle \]

\[ \rho \quad \text{U} \]

With \( \gamma = Tr\{U \rho\} \),

\[ \rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1|\gamma^* + |1\rangle\langle 0|\gamma + |1\rangle\langle 1|) \]
Trace circuit

\[ |0\rangle \langle 0| \xrightarrow{H} \rho \xrightarrow{U} \rho_1 \xrightarrow{H} \langle \sigma_k \rangle \]

With \( \gamma = Tr\{U \rho\} \),

\[
\rho_1 = \frac{1}{2}(|0\rangle \langle 0| + |0\rangle \langle 1| \gamma^* + |1\rangle \langle 0| \gamma + |1\rangle \langle 1|)
\]

\[
= \frac{1}{2} \begin{pmatrix} 1 & \gamma^* \\ \gamma & 1 \end{pmatrix}
\]
Trace circuit

With \( \gamma = Tr\{U \rho\} \),

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\]

\[
= \frac{1}{2} \begin{pmatrix}
1 & \gamma^* \\
\gamma & 1
\end{pmatrix}
\]

\[
\rightarrow \frac{1}{2} \begin{pmatrix}
1 + Re\{\gamma\} & iIm\{\gamma\} \\
-iIm\{\gamma\} & 1 - Re\{\gamma\}
\end{pmatrix}
\]
If $U$ can be implemented efficiently and prepared efficiently, we can evaluate $\text{Tr}_f U$ within accuracy in time $\text{poly}(n) = 2^n$. 

$$\rho'_1 = \frac{1}{2} \begin{pmatrix} 1 + \text{Re}\{\gamma\} & i\text{Im}\{\gamma\} \\ -i\text{Im}\{\gamma\} & 1 - \text{Re}\{\gamma\} \end{pmatrix}$$
If $U$ can be implemented efficiently and prepared efficiently, we can evaluate $\Tr f U g$ within accuracy in time $\text{poly}(n) = 2^n$. 

$k'=z: \langle \sigma_z \rangle = \Tr \{ \rho_1 \sigma_z \} = \Re \{ \gamma \} = \Re \{ \Tr \{ \rho U \} \}$

$k'=y: \langle \sigma_y \rangle = \Tr \{ \rho_1 \sigma_y \} = \Im \{ \gamma \} = \Im \{ \Tr \{ \rho U \} \}$
If $U$ can be implemented efficiently and $\rho$ prepared efficiently, we can evaluate $Tr\{\rho U\}$ within accuracy $\delta$ in time $poly(n)/\delta^2$. 
Trace circuit

Trace of the product of a density matrix and a unitary transformation: $Tr\{\rho U\}$

Can we get useful partial information about $U$?

(Tomography is exponentially hard.)
Trace circuit

\[ Tr\{ \rho U \} \]

Control \( U \) and learn about \( \rho \): quantum state tomography.
Trace circuit

\[ Tr \{ \rho U \} \]

- Control \( U \) and learn about \( \rho \): quantum state tomography.
- Control \( \rho \) and learn about \( U \): scattering or quantum process tomography.

Can we get useful partial information about \( U \)?

(Tomography is exponentially hard.)
$Tr\{\rho U\}$

- Control $U$ and learn about $\rho$: quantum state tomography.
- Control $\rho$ and learn about $U$: scattering or quantum process tomography.

Can we get useful *partial* information about $U$? (Tomography is exponentially hard.)
Let’s see what we can do in the DQC1 model.

\[ |0\rangle \langle 0| \quad \text{H} \quad \text{H} \quad \langle \sigma_k \rangle \]

\[ \frac{1}{N} 1 \]

\[ U \]
Let’s see what we can do in the DQC1 model.

\[ |0\rangle \langle 0| \quad \text{H} \quad \text{H} \quad \langle \sigma_k \rangle \]

\[ \frac{1}{N} \text{Tr}\{U\} = \frac{1}{N} \sum_{j=1}^{N} \lambda_j \]
Quantum control

QIP is about manipulating complex quantum systems.

Control a quantum system = exploit its symmetries (QEC, NSS, etc.)

Does my system possess any symmetries?

Systems with as many symmetries as degrees of freedom are integrable.

Chaotic systems possess no or very few symmetries: They are hard to control.
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Looking for symmetries

Resonances of Thorium 232

Looking for symmetries

List of resonances

| E₀ (MeV) | ΔE₀ (MeV) | J₀ | J₁ | A₀ | A₁ | T₀ | T₁ | C₀ | C₁ | A₀ (MeV) | A₁ (MeV) | T₀ (MeV) | T₁ (MeV) | C₀ (MeV) | C₁ (MeV) |
|---------|-----------|----|----|----|----|----|----|----|----|----|---------|---------|---------|---------|---------|---------|
| 1.28     | 0.00      | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    |
| 2.56     | 0.00      | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    |
| 3.84     | 0.00      | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    |
| 5.12     | 0.00      | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    |

*Δ and A with E₀, J₀, and J₁, T₀, and T₁, C₀, and C₁ are from the work of Jones (1973).

Level spacing

Fig. 10. Histogram of the observed distribution of nearest-neighbor level spacings \( x = D/D₀ \) for Th. The three theoretical curves correspond respectively to random, orthogonal, \( (\beta = 1) \) and unitary \( (\beta = 2) \). The orthogonal \( (\beta = 1) \) curve is the favored theoretical distribution.
Looking for symmetries

Wigner’s idea: The main characteristics of the spectrum can be reproduced by random matrices which possess the same symmetries of the system.
Looking for symmetries

Wigner’s idea: The main characteristics of the spectrum can be reproduced by random matrices which possess the same symmetries of the system.

- No symmetry = Level repulsion
Looking for symmetries

Symmetries = Block diagonal

$$U = \exp\left\{ \frac{-iHt}{\hbar} \right\} = \begin{pmatrix}
U_1 \\

\ddots \\

U_d
\end{pmatrix}$$
Looking for symmetries

Symmetries = Block diagonal

\[ U = \exp\left\{ \frac{-iHt}{\hbar} \right\} = \begin{pmatrix} U_1 & & \\ & U_2 & \\ & & \ddots \\ & & & U_d \end{pmatrix} \]

\[ U_1 + U_2 + \ldots = U \]

\[ Pr(d) \]
Looking for symmetries

\[ \frac{1}{N} \text{Tr}\{U\} = \frac{1}{N} \sum_j e^{i\phi_j} \]

\[ = \sum_j \tilde{v}_j \]
Looking for symmetries

\[ \frac{1}{N} Tr\{U\} = \frac{1}{N} \sum_j e^{i\phi_j} \]

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Looking for symmetries

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Looking for symmetries

\[ \frac{1}{N} Tr \{ U \} = \frac{1}{N} \sum_j e^{i \phi_j} \]
\[ = \sum_j v_j \]

With symmetries (regular): \[ \frac{1}{N} Tr \{ U \} = \frac{1}{N} \sqrt{N} = \frac{1}{\sqrt{N}} \]
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- No symmetries (chaotic): \( \frac{1}{N} Tr\{U\} = \frac{1}{N} O(1) = \frac{1}{N} \)

Requires accuracy \( \frac{1}{\sqrt{N}} \) so overhead is \( N \).

Fidelity decay

Quantum map $U$, e.g. $U = \exp\{-iHt\}$
Fidelity decay

Quantum map $U$, e.g. $U = \exp\{-iHt\}$

Perturbed quantum map $U_p$, e.g.

$$U_p = U \exp\{-i\delta V\}$$
Fidelity decay

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$F_n(\psi) = \left| \langle \psi | (U^n)^\dagger U^n_p | \psi \rangle \right|^2$
Fidelity decay

Quantum map $U$, e.g. $U = \exp\{-iHt\}$

Perturbed quantum map $U_p$, e.g.

$$U_p = U \left\{ \exp\{-i\delta V\} \right\}_K$$

$$F_n(\psi) = \left| \langle \psi \mid (U^n)^\dagger U_p^n \mid \psi \rangle \right|^2$$

$$\overline{F_n} = \int F_n(\psi) d\psi = \frac{\left| \text{Tr}\{(U^n)^\dagger U_p^n\} \right|^2 + N}{N^2 + N}$$
Fidelity decay

Peres’ conjecture: $F_n(\psi)$ is a signature of quantum chaos (dynamical instability).

Chaotic systems have an exponential decay $\exp(-\alpha n)$. Regular systems have a polynomial decay $1=\text{poly}(n)$. Fidelity decay measures the relative randomness of $U$ and $K$. Universal response to perturbation. "Simple" perturbations produce good signatures of quantum chaos.

J. Emerson, Y.S. Weinstein, S. Lloyd, and D.G. Cory, 2002

MIT's Independent Activities Period, January 2004 -- p.22
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- Fidelity decay measures the relative randomness of $U$ and $K$.
  - Universal response to perturbation.
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\[ |0\rangle\langle 0| \quad H \quad H \quad \langle \sigma_k \rangle \]

\[ \rho \quad U \]

Trace circuit

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\[ k = y : \langle \sigma_y \rangle = \text{Im} [\text{Tr}\{\rho U\}] \]

\[ k = z : \langle \sigma_z \rangle = \text{Re} [\text{Tr}\{\rho U\}] \]

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\[ k = y : \langle \sigma_y \rangle = \text{Im} \left[ \text{Tr} \{ \rho U \} \right] \]

\[ k = z : \langle \sigma_z \rangle = \text{Re} \left[ \text{Tr} \{ \rho U \} \right] \]

\[ F_n = \frac{|\text{Tr}\{ (U^n)^\dagger U^n_p \}|^2 + N}{N^2 + N}, \quad U_p = U K \]
Since the $U$'s and $U^\dagger$'s annihilate each other when the $K$'s are absent. Since the $U^\dagger$'s are made after the final measurement. Measure $F_n$ within accuracy with error probability $p$ in time $n \cdot O(\log(1/p^2)T(n))$.
Since the $U$’s and $U^\dagger$’s annihilate each others when the $K$’s are absent.
Since the $U$’s and $U^\dagger$’s annihilate each others when the $K$’s are absent.

Since the $U^\dagger$ are made after the final measurement.
Quantum circuit

\[
|0\rangle \xrightarrow{\text{H}} U \xrightarrow{\text{K}} U \xrightarrow{\text{K}} \ldots \xrightarrow{\text{U}} \xrightarrow{\text{K}} \langle \sigma_k \rangle
\]

Since the $U$'s and $U^\dagger$'s annihilate each other when the $K$'s are absent.

Since the $U^\dagger$ are made after the final measurement.

Measure $\overline{F_n}$ within accuracy $\epsilon$ with error probability $p$ in time

\[
nO \left( \frac{\log(1/p)}{\epsilon^2} \right) T(n)
\]
Decoherence

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It arises from a coupling between the system and its environment (uncontrolled degrees of freedom).

The stronger the coupling to the environment, the faster decoherence takes place.

\[
\alpha|0\rangle + \beta|1\rangle \xrightarrow{S} \alpha|00\rangle + \beta|11\rangle
\]

\[
\rho_S \rightarrow |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|
\]
The decoherence rate is governed by the average fidelity decay rate of the environment!

\[ \rho_0 \quad \frac{1}{N} \quad U \quad K \quad U \quad K \quad \ldots \quad U \quad K \quad \rho_n \]
Decoherence

\begin{align*}
\rho_0 &= \begin{pmatrix}
|\alpha|^2 & \alpha\beta^* \\
\alpha^*\beta & |\beta|^2
\end{pmatrix} \quad \Rightarrow \quad \rho_n = \begin{pmatrix}
|\alpha|^2 & \alpha\beta^*\gamma(n) \\
\alpha^*\beta\gamma(n)^* & |\beta|^2
\end{pmatrix}
\end{align*}
Decoherence

\[ \rho_0 = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \Rightarrow \rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\gamma(n)^* \\ \alpha^*\beta\gamma(n) & |\beta|^2 \end{pmatrix} \]

\[ |\gamma(n)| = \left\{ \overline{F_n} \right\}^{1/2} \]: The decoherence rate is governed by the average fidelity decay rate of the environment!
The final state is

$$|\psi\rangle = \frac{1}{N} \left( \alpha |0\rangle + \beta |1\rangle \right)$$

where $\alpha$ and $\beta$ are complex numbers. The $U$ and $K$ operations are unitary and projective measurements, respectively, in the quantum circuit.
The final state is

\[ \rho_k = \frac{1}{N} \sum_j |\alpha_j\rangle \langle \alpha_j| \otimes |\phi_j\rangle \langle \phi_j| \]

where \( |\alpha_j\rangle = \alpha|0\rangle + \beta e^{is_j}|1\rangle \) and the \( e^{is_j} \) are the eigenvalues of \( S = U_k P_k \ldots U_2 P_2 U_1 P_1 U_1^\dagger U_2^\dagger \ldots U_k^\dagger \).
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Quantum probe

\[ \frac{1}{N} \sum_{i,j=1}^{N} H(i;j) = \rho_0 \]
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\[
\rho_0 \quad \frac{1}{N} \quad U \quad K \quad U \quad K \quad \ldots \quad U \quad K
\]

\[
H = \sum_{j>1} J_{1j} Z_1 Z_j + \underbrace{H(i, j \neq 1)}_{\text{perturbation}} + \text{Unknown } U
\]