Operator Quantum Error Correction
An Overview

David Poulin
School of Physical Sciences
The University of Queensland

McGill University, March 2006
<table>
<thead>
<tr>
<th>Outline</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong></td>
<td><strong>Introduction</strong></td>
</tr>
<tr>
<td></td>
<td>- Context</td>
</tr>
<tr>
<td></td>
<td>- Results summary</td>
</tr>
<tr>
<td><strong>2</strong></td>
<td><strong>Quantum error correction</strong></td>
</tr>
<tr>
<td></td>
<td>- Definitions</td>
</tr>
<tr>
<td></td>
<td>- Stabilizer formalism</td>
</tr>
<tr>
<td><strong>3</strong></td>
<td><strong>Noiseless subsystems</strong></td>
</tr>
<tr>
<td></td>
<td>- Definition</td>
</tr>
<tr>
<td></td>
<td>- Generalization</td>
</tr>
<tr>
<td><strong>4</strong></td>
<td><strong>Operator quantum error correction</strong></td>
</tr>
<tr>
<td></td>
<td>- Definition</td>
</tr>
<tr>
<td></td>
<td>- Conditions</td>
</tr>
<tr>
<td></td>
<td>- Stabilizer formalism</td>
</tr>
<tr>
<td><strong>5</strong></td>
<td><strong>Summary</strong></td>
</tr>
</tbody>
</table>
1. Introduction
   - Context
     - Results summary

2. Quantum error correction
   - Definitions
   - Stabilizer formalism

3. Noiseless subsystems
   - Definition
   - Generalization

4. Operator quantum error correction
   - Definition
   - Conditions
   - Stabilizer formalism

5. Summary
Research Interests

My research interests can be roughly divided into two categories:

- Quantum information theory as an end.
- Quantum information theory applied to fundamental physics.
**Measurement problem**


**Constraint systems and quantum gravity**

Fundamental physics

**Measurement problem**

**Constraint systems and quantum gravity**
Quantum Information

- **Quantum algorithms for simulation**

- **Quantum key distribution**

- **Quantum error correction**
Quantum Information

Quantum algorithms for simulation

Quantum key distribution

Quantum error correction
Quantum Information

Quantum algorithms for simulation

Quantum key distribution

Quantum error correction
For those who want to sleep...

With the right modifications of existing concepts, operator quantum error correction (OQEC) can be defined trivially.

- **Quantum Error Correction**
  - Active recovery procedure.
  - Code subspace $H = C \oplus C^\perp$.
  - $\mathcal{E}$ is correctable iff exists recovery $(\mathcal{R} \circ \mathcal{E})(\rho) = \rho \quad \forall \rho = P_C \rho P_C$.

- **Noiseless subsystem**
  - Passive, error avoiding.
  - Code subsystem $H = (A \otimes B) \oplus C^\perp$.
  - $A$ is noiseless for $\mathcal{E}$ iff $\mathcal{E}(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B \quad \forall \rho^A$.

- **Operator quantum error correction**
  - Active recovery procedure.
  - Code subsystem $H = (A \otimes B) \oplus C^\perp$.
  - $(\mathcal{R} \circ \mathcal{E})(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B \quad \forall \rho^A$.
  - Non Abelian version of QEC.
With the right modifications of existing concepts, operator quantum error correction (OQEC) can be defined trivially.

- **Quantum Error Correction**
  - Active recovery procedure.
  - Code subspace $H = C \oplus C^\perp$.
  - $\mathcal{E}$ is correctable iff exists recovery $(\mathcal{R} \circ \mathcal{E})(\rho) = \rho$ — $\forall \rho = P_C \rho P_C$.

- **Noiseless subsystem**
  - Passive, error avoiding.
  - Code subsystem $H = (A \otimes B) \oplus C^\perp$
  - $A$ is noiseless for $\mathcal{E}$ iff $\mathcal{E}(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B$ — $\forall \rho^A$.

- **Operator quantum error correction**
  - Active recovery procedure.
  - Code subsystem $H = (A \otimes B) \oplus C^\perp$.
  - $(\mathcal{R} \circ \mathcal{E})(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B$ — $\forall \rho^A$.
  - Non Abelian version of QEC.
For those who want to sleep...

With the right modifications of existing concepts, operator quantum error correction (OQEC) can be defined trivially.

- **Quantum Error Correction**
  - Active recovery procedure.
  - Code subspace \( H = C \oplus C^\perp \).
  - \( \mathcal{E} \) is correctable iff exists recovery \((\mathcal{R} \circ \mathcal{E})(\rho) = \rho \quad \forall \rho = P_C \rho P_C\).

- **Noiseless subsystem**
  - Passive, error avoiding.
  - Code subsystem \( H = (A \otimes B) \oplus C^\perp \)
  - \( A \) is noiseless for \( \mathcal{E} \) iff \( \mathcal{E}(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B \quad \forall \rho^A \).

- **Operator quantum error correction**
  - Active recovery procedure.
  - Code subspace \( H = (A \otimes B) \oplus C^\perp \).
  - \( (\mathcal{R} \circ \mathcal{E})(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B \quad \forall \rho^A \).
  - Non Abelian version of QEC.
Outline

1. Introduction
   - Context
   - Results summary

2. Quantum error correction
   - Definitions
   - Stabilizer formalism

3. Noiseless subsystems
   - Definition
   - Generalization

4. Operator quantum error correction
   - Definition
   - Conditions
   - Stabilizer formalism

5. Summary
A physical map $\mathcal{E} : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ acting on a finite quantum system can always be written as

$$
\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger
$$

with $\sum_a E_a^\dagger E_a = 1\!1$ to ensure trace preservation.

- $\{E_a\}$ and $\{F_b\}$ describe the same map iff $E_a = \sum_b u_{ab} F_b$ for some (padded) unitary matrix $u$.
- $\mathcal{E}$ is called unital if $\mathcal{E}(1\!1) = 1\!1$ (dual is trace preserving).
A physical map $\mathcal{E} : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ acting on a finite quantum system can always be written as

$$\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$$

with $\sum_a E_a^\dagger E_a = 1_1$ to ensure trace preservation.

- $\{E_a\}$ and $\{F_b\}$ describe the same map iff $E_a = \sum_b u_{ab} F_b$ for some (padded) unitary matrix $u$.
- $\mathcal{E}$ is called unital if $\mathcal{E}(1_1) = 1_1$ (dual is trace preserving).
A physical map $\mathcal{E} : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ acting on a finite quantum system can always be written as

$$\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$$

with $\sum_a E_a^\dagger E_a = 11$ to ensure trace preservation.

- $\{E_a\}$ and $\{F_b\}$ describe the same map iff $E_a = \sum_b u_{ab} F_b$ for some (padded) unitary matrix $u$.
- $\mathcal{E}$ is called unital if $\mathcal{E}(11) = 11$ (dual is trace preserving).
Correctable

Definition

Given a decomposition $H = C \oplus C^\perp$, the map $\mathcal{E}$ is correctable on subspace $C$ iff there exists a recovery map $\mathcal{R}$ such that

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho \quad \forall \rho \in \mathcal{B}(C).$$

Theorem (BDSW96,KL97)

The map $\mathcal{E}$ is correctable iff $P_CP_a^\dagger E_bP_C = \lambda_{ab}P_C$ for all pairs $E_a, E_b$, where $\lambda$'s are scalars.

- This definition is independent of the choice of Kraus operators.
Correctable

**Definition**

Given a decomposition $H = C \oplus C^\perp$, the map $\mathcal{E}$ is correctable on subspace $C$ iff there exists a recovery map $\mathcal{R}$ such that

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho \quad \forall \rho \in \mathcal{B}(C).$$

**Theorem (BDSW96,KL97)**

The map $\mathcal{E}$ is correctable iff $P_C E_a^\dagger E_b P_C = \lambda_{ab} P_C$ for all pairs $E_a, E_b$, where $\lambda$’s are scalars.

- This definition is independent of the choice of Kraus operators.
Outline

1. Introduction
   - Context
   - Results summary

2. Quantum error correction
   - Definitions
   - Stabilizer formalism

3. Noiseless subsystems
   - Definition
   - Generalization

4. Operator quantum error correction
   - Definition
   - Conditions
   - Stabilizer formalism

5. Summary
Pauli group

- Single qubit Pauli operators $1\mathbb{I}$, $X$, $Y$ and $Z$.
  - $X^2 = Y^2 = Z^2 = 1\mathbb{I}$.
  - $PQ = \pm QP$: + if $P$ or $Q = 1\mathbb{I}$ or if $P = Q$; – otherwise.
- Denote $X_j = 1\mathbb{I} \otimes \ldots \otimes 1\mathbb{I} \otimes X \otimes 1\mathbb{I} \otimes \ldots \otimes 1\mathbb{I}$, idem for $Y$ and $Z$.
- $n$-qubit Pauli group: $\mathcal{P}_n = \{\eta P_1 P_2 \ldots P_n\}$, with $P_j \in \{1\mathbb{I}, X, Y, Z\}$ and $\eta \in \{\pm 1, \pm i\}$.
  - $P^2 = \pm 1\mathbb{I}$ for all $P \in \mathcal{P}_n$.
  - $PQ = \pm QP$: count the parity of the anti-commuting single-qubit Pauli operators.
  - $\mathcal{P}_n$ is generated under multiplication by the $X_j$ and the $Z_j$, i.e. $\mathcal{P}_n = \langle i, X_1, Z_1, \ldots X_n, Z_n \rangle$. 
Pauli group

- Single qubit Pauli operators $1\mathbb{I}$, $X$, $Y$ and $Z$.
  - $X^2 = Y^2 = Z^2 = 1\mathbb{I}$.
  - $PQ = \pm QP$: + if $P$ or $Q = 1\mathbb{I}$ or if $P = Q$; – otherwise.
  - Denote $X_j = 1\mathbb{I} \otimes \ldots \otimes 1\mathbb{I} \otimes X \otimes 1\mathbb{I} \otimes \ldots \otimes 1\mathbb{I}$, idem for $Y$ and $Z$.

- $n$-qubit Pauli group: $\mathcal{P}_n = \{\eta P_1 P_2 \ldots P_n\}$, with $P_j \in \{1\mathbb{I}, X, Y, Z\}$ and $\eta \in \{\pm 1, \pm i\}$.
  - $P^2 = \pm 1\mathbb{I}$ for all $P \in \mathcal{P}_n$.
  - $PQ = \pm QP$: count the parity of the anti-commuting single-qubit Pauli operators.
  - $\mathcal{P}_n$ is generated under multiplication by the $X_j$ and the $Z_j$, i.e. $\mathcal{P}_n = \langle i, X_1, Z_1, \ldots X_n, Z_n \rangle$. 
Pauli group

- Single qubit Pauli operators $\mathbb{1}$, $X$, $Y$ and $Z$.
  - $X^2 = Y^2 = Z^2 = \mathbb{1}$.
  - $PQ = \pm QP$: + if $P$ or $Q = \mathbb{1}$ or if $P = Q$; − otherwise.

- Denote $X_j = \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes X \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$, idem for $Y$ and $Z$.

- $n$-qubit Pauli group: $\mathcal{P}_n = \{ \eta P_1 P_2 \ldots P_n \}$, with $P_j \in \{ \mathbb{1}, X, Y, Z \}$ and $\eta \in \{ \pm 1, \pm i \}$.
  - $P^2 = \pm \mathbb{1}$ for all $P \in \mathcal{P}_n$.
  - $PQ = \pm QP$: count the parity of the anti-commuting single-qubit Pauli operators.
  - $\mathcal{P}_n$ is generated under multiplication by the $X_j$ and the $Z_j$, i.e. $\mathcal{P}_n = \langle i, X_1, Z_1, \ldots X_n, Z_n \rangle$. 

Quantum error correction

Stabilizer formalism

Pauli group

- Single qubit Pauli operators $\mathbb{1}$, $X$, $Y$ and $Z$.
  - $X^2 = Y^2 = Z^2 = \mathbb{1}$.
  - $PQ = \pm QP$: + if $P$ or $Q = \mathbb{1}$ or if $P = Q$; – otherwise.
- Denote $X_j = \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes X \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$, idem for $Y$ and $Z$.
- $n$-qubit Pauli group: $\mathcal{P}_n = \{ \eta P_1 P_2 \ldots P_n \}$, with $P_j \in \{ \mathbb{1}, X, Y, Z \}$ and $\eta \in \{ \pm 1, \pm i \}$.
  - $P^2 = \pm \mathbb{1}$ for all $P \in \mathcal{P}_n$.
  - $PQ = \pm QP$: count the parity of the anti-commuting single-qubit Pauli operators.
- $\mathcal{P}_n$ is generated under multiplication by the $X_j$ and the $Z_j$, i.e. $\mathcal{P}_n = \langle i, X_1, Z_1, \ldots X_n, Z_n \rangle$. 
Pauli group

- Single qubit Pauli operators $\mathbb{1}, X, Y$ and $Z$.
  - $X^2 = Y^2 = Z^2 = \mathbb{1}$.
  - $PQ = \pm QP$: $+$ if $P$ or $Q = \mathbb{1}$ or if $P = Q$; $-$ otherwise.
- Denote $X_j = \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes X \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$, idem for $Y$ and $Z$.
- $n$-qubit Pauli group: $\mathcal{P}_n = \{ \eta P_1 P_2 \ldots P_n \}$, with $P_j \in \{ \mathbb{1}, X, Y, Z \}$ and $\eta \in \{ \pm 1, \pm i \}$.
  - $P^2 = \pm \mathbb{1}$ for all $P \in \mathcal{P}_n$.
  - $PQ = \pm QP$: count the parity of the anti-commuting single-qubit Pauli operators.
  - $\mathcal{P}_n$ is generated under multiplication by the $X_j$ and the $Z_j$, i.e. $\mathcal{P}_n = \langle i, X_1, Z_1, \ldots X_n, Z_n \rangle$. 
Pauli group

- Single qubit Pauli operators $1\mathbb{I}$, $X$, $Y$ and $Z$.
  - $X^2 = Y^2 = Z^2 = 1\mathbb{I}$.
  - $PQ = \pm QP$: $+$ if $P$ or $Q = 1\mathbb{I}$ or if $P = Q$; $-$ otherwise.
- Denote $X_j = 1\mathbb{I} \otimes \ldots \otimes 1\mathbb{I} \otimes X \otimes 1\mathbb{I} \otimes \ldots \otimes 1\mathbb{I}$, idem for $Y$ and $Z$.
- $n$-qubit Pauli group: $\mathcal{P}_n = \{\eta P_1 P_2 \ldots P_n\}$, with $P_j \in \{1\mathbb{I}, X, Y, Z\}$ and $\eta \in \{\pm 1, \pm i\}$.
  - $P^2 = \pm 1\mathbb{I}$ for all $P \in \mathcal{P}_n$.
  - $PQ = \pm QP$: count the parity of the anti-commuting single-qubit Pauli operators.
  - $\mathcal{P}_n$ is generated under multiplication by the $X_j$ and the $Z_j$, i.e. $\mathcal{P}_n = \langle i, X_1, Z_1, \ldots X_n, Z_n \rangle$. 
Pauli group

- Single qubit Pauli operators $1\l, X, Y$ and $Z$.
  - $X^2 = Y^2 = Z^2 = 1\l$.
  - $PQ = \pm QP$: + if $P$ or $Q = 1\l$ or if $P = Q$; − otherwise.
- Denote $X_j = 1\l \otimes \ldots \otimes 1\l \otimes X \otimes 1\l \otimes \ldots \otimes 1\l$, idem for $Y$ and $Z$.
- $n$-qubit Pauli group: $\mathcal{P}_n = \{\eta P_1 P_2 \ldots P_n\}$, with $P_j \in \{1\l, X, Y, Z\}$ and $\eta \in \{\pm 1, \pm i\}$.
  - $P^2 = \pm 1\l$ for all $P \in \mathcal{P}_n$.
  - $PQ = \pm QP$: count the parity of the anti-commuting single-qubit Pauli operators.
- $\mathcal{P}_n$ is generated under multiplication by the $X_j$ and the $Z_j$, i.e. $\mathcal{P}_n = \langle i, X_1, Z_1, \ldots X_n, Z_n \rangle$. 
Pauli group

- Single qubit Pauli operators $\mathbb{1}$, $X$, $Y$ and $Z$.
  - $X^2 = Y^2 = Z^2 = \mathbb{1}$.
  - $PQ = \pm QP$: $+$ if $P$ or $Q = \mathbb{1}$ or if $P = Q$; $-$ otherwise.

- Denote $X_j = \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes X \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$, idem for $Y$ and $Z$.

- $n$-qubit Pauli group: $\mathcal{P}_n = \{ \eta P_1 P_2 \ldots P_n \}$, with $P_j \in \{ \mathbb{1}, X, Y, Z \}$ and $\eta \in \{ \pm 1, \pm i \}$.
  - $P^2 = \pm \mathbb{1}$ for all $P \in \mathcal{P}_n$.
  - $PQ = \pm QP$: count the parity of the anti-commuting single-qubit Pauli operators.
  - $\mathcal{P}_n$ is generated under multiplication by the $X_j$ and the $Z_j$, i.e. $\mathcal{P}_n = \langle i, X_1, Z_1, \ldots X_n, Z_n \rangle$. 
Clifford operations

- Unitary matrices that acts as a permutation on the Pauli group: for all $P \in \mathcal{P}_n$, we have $UPU^\dagger = P'$ for some $P' \in \mathcal{P}_n$.
- Can be realized with CNOTS, HADAMARD, and PHASE.
- Can be simulated efficiently classically.

Example

\[
\begin{align*}
\text{CNOT} \cdot X11 \cdot \text{CNOT}^\dagger &= XX \\
\text{CNOT} \cdot 11X \cdot \text{CNOT}^\dagger &= 11X \\
\text{CNOT} \cdot Z11 \cdot \text{CNOT}^\dagger &= Z11 \\
\text{CNOT} \cdot 11Z \cdot \text{CNOT}^\dagger &= ZZ
\end{align*}
\]
Clifford operations

- Unitary matrices that acts as a permutation on the Pauli group: for all $P \in \mathcal{P}_n$, we have $U P U^\dagger = P'$ for some $P' \in \mathcal{P}_n$.
- Can be realized with CNOTS, HADAMARD, and PHASE.
- Can be simulated efficiently classically.

Example

\[
\begin{align*}
\text{CNOT} \cdot X \text{1l} \cdot \text{CNOT}^\dagger &= XX \\
\text{CNOT} \cdot \text{1l}X \cdot \text{CNOT}^\dagger &= 1lX \\
\text{CNOT} \cdot Z \text{1l} \cdot \text{CNOT}^\dagger &= Z1l \\
\text{CNOT} \cdot 1lZ \cdot \text{CNOT}^\dagger &= ZZ
\end{align*}
\]
Clifford operations

- Unitary matrices that acts as a permutation on the Pauli group: for all $P \in \mathcal{P}_n$, we have $UPU^\dagger = P'$ for some $P' \in \mathcal{P}_n$.
- Can be realized with CNOTS, HADAMARD, and PHASE.
- Can be simulated efficiently classically.

Example

\[
\begin{align*}
\text{CNOT} \cdot X \mathbb{1} \cdot \text{CNOT}^\dagger &= XX \\
\text{CNOT} \cdot \mathbb{1} \cdot X \cdot \text{CNOT}^\dagger &= 11X \\
\text{CNOT} \cdot Z \mathbb{1} \cdot \text{CNOT}^\dagger &= Z \mathbb{1} \\
\text{CNOT} \cdot \mathbb{1} \cdot Z \cdot \text{CNOT}^\dagger &= ZZ
\end{align*}
\]
Clifford operations

- Unitary matrices that acts as a permutation on the Pauli group: for all $P \in \mathcal{P}_n$, we have $UPU^\dagger = P'$ for some $P' \in \mathcal{P}_n$.
- Can be realized with CNOTS, HADAMARD, and PHASE.
- Can be simulated efficiently classically.

**Example**

\[
\begin{align*}
\text{CNOT} \cdot X 1\bar{1} \cdot \text{CNOT}^\dagger &= XX \\
\text{CNOT} \cdot 1\bar{1}X \cdot \text{CNOT}^\dagger &= 1\bar{1}X \\
\text{CNOT} \cdot Z 1\bar{1} \cdot \text{CNOT}^\dagger &= Z 1\bar{1} \\
\text{CNOT} \cdot 1\bar{1}Z \cdot \text{CNOT}^\dagger &= ZZ
\end{align*}
\]
A code is specified by a subspace $C$ of $(\mathbb{C}^2)^n$.

Specify this subspace by constraints: $S_j|\psi\rangle = |\psi\rangle$ for $j = 1, \ldots, n - k$, $S_j \in \mathcal{P}_n$.

- The stabilizer generators $S_j$ must commute.
- They generate an Abelian subgroup of $\mathcal{P}_n$ called the stabilizer $S = \langle S_1, \ldots, S_{n-k} \rangle$.
- We must have $\{i, -i, -1\} \notin S$.
- The dimension of $C$ is $2^k$, it encodes $k$ qubits.

**Example**

With $S = \langle ZZ11, 11ZZ \rangle$, we get a $3 - 2 = 1$ qubit subspace spanned by the states $|000\rangle$ and $|111\rangle$. 
A code is specified by a subspace $C$ of $(\mathbb{C}^2)^n$. Specify this subspace by constraints: $S_j|\psi\rangle = |\psi\rangle$ for $j = 1, \ldots, n-k$, $S_j \in \mathcal{P}_n$.
- The stabilizer generators $S_j$ must commute.
- They generate an Abelian subgroup of $\mathcal{P}_n$ called the stabilizer $S = \langle S_1, \ldots S_{n-k} \rangle$.
- We must have $\{i, -i, -1\} \notin S$.
- The dimension of $C$ is $2^k$, it encodes $k$ qubits.

**Example**

With $S = \langle ZZ11, 11ZZ \rangle$, we get a $3 - 2 = 1$ qubit subspace spanned by the states $|000\rangle$ and $|111\rangle$. 
A code is specified by a subspace $C$ of $(\mathbb{C}^2)^n$.

Specify this subspace by constraints: $S_j|\psi\rangle = |\psi\rangle$ for $j = 1, \ldots, n - k$, $S_j \in \mathcal{P}_n$.

- The stabilizer generators $S_j$ must commute.
- They generate an Abelian subgroup of $\mathcal{P}_n$ called the stabilizer $S = \langle S_1, \ldots, S_{n-k} \rangle$.
- We must have $\{i, -i, -1\} \notin S$.
- The dimension of $C$ is $2^k$, it encodes $k$ qubits.

**Example**

With $S = \langle ZZ1\mathbb{I}, \mathbb{I}ZZ \rangle$, we get a $3 - 2 = 1$ qubit subspace spanned by the states $|000\rangle$ and $|111\rangle$. 
A code is specified by a subspace $C$ of $(\mathbb{C}^2)^n$.

Specify this subspace by constraints: $S_j|\psi\rangle = |\psi\rangle$ for $j = 1, \ldots, n - k$, $S_j \in \mathcal{P}_n$.

- The stabilizer generators $S_j$ must commute.
- They generate an Abelian subgroup of $\mathcal{P}_n$ called the stabilizer $S = \langle S_1, \ldots S_{n-k}\rangle$.
- We must have $\{i, -i, -1\} \notin S$.
- The dimension of $C$ is $2^k$, it encodes $k$ qubits.

Example

With $S = \langle ZZ11, 11ZZ\rangle$, we get a $3 - 2 = 1$ qubit subspace spanned by the states $|000\rangle$ and $|111\rangle$. 

David Poulin (Queensland)  Operator Quantum Error Correction  Montréal 2006  15 / 38
A code is specified by a subspace $C$ of $(\mathbb{C}^2)^n$.

Specify this subspace by constraints: $S_j |\psi\rangle = |\psi\rangle$ for $j = 1, \ldots n - k$, $S_j \in \mathcal{P}_n$.

- The stabilizer generators $S_j$ must commute.
- They generate an Abelian subgroup of $\mathcal{P}_n$ called the stabilizer $S = \langle S_1, \ldots S_{n-k} \rangle$.
- We must have $\{i, -i, -1\} \notin S$.
- The dimension of $C$ is $2^k$, it encodes $k$ qubits.

Example

With $S = \langle ZZ1\bar{1}, 1\bar{1}ZZ \rangle$, we get a $3 - 2 = 1$ qubit subspace spanned by the states $|000\rangle$ and $|111\rangle$. 
A code is specified by a subspace $C$ of $(\mathbb{C}^2)^n$.

Specify this subspace by constraints: $S_j|\psi\rangle = |\psi\rangle$ for $j = 1, \ldots, n - k$, $S_j \in \mathcal{P}_n$.

- The stabilizer generators $S_j$ must commute.
- They generate an Abelian subgroup of $\mathcal{P}_n$ called the stabilizer $S = \langle S_1, \ldots, S_{n-k} \rangle$.
- We must have $\{i, -i, -1\} \notin S$.
- The dimension of $C$ is $2^k$, it encodes $k$ qubits.

**Example**

With $S = \langle ZZ1l, 1lZZ \rangle$, we get a $3 - 2 = 1$ qubit subspace spanned by the states $|000\rangle$ and $|111\rangle$. 
A code is specified by a subspace \( C \) of \( (\mathbb{C}^2)^n \).

Specify this subspace by constraints: \( S_j|\psi\rangle = |\psi\rangle \) for \( j = 1, \ldots n - k, S_j \in \mathcal{P}_n \).

- The stabilizer generators \( S_j \) must commute.
- They generate an Abelian subgroup of \( \mathcal{P}_n \) called the stabilizer \( S = \langle S_1, \ldots S_{n-k} \rangle \).
- We must have \( \{i, -i, -1\} \not\in S \).
- The dimension of \( C \) is \( 2^k \), it encodes \( k \) qubits.

**Example**

With \( S = \langle ZZ11, 11ZZ \rangle \), we get a \( 3 - 2 = 1 \) qubit subspace spanned by the states \( |000\rangle \) and \( |111\rangle \).
We only consider errors in $\mathcal{P}_n$ since they form a basis for all matrix: for $E = \sum_a \alpha_a P_a$

$$E|\psi\rangle = \sum_a \alpha_a P_a|\psi\rangle$$

Measurement of $S_j$ — called syndrome — indicates if the system is in the code space $C$.

Given syndrome $(m_1, \ldots, m_{n-k}) \in \{-1, 1\}^{n-k}$, we know that an error $E$ occurred with $ES_j = m_j S_j E$ since

$$S_j E|\psi\rangle = \pm ES_j |\psi\rangle = \pm E|\psi\rangle.$$
We only consider errors in $\mathcal{P}_n$ since they form a basis for all matrix: for $E = \sum_a \alpha_a P_a$

$$E|\psi\rangle = \sum_a \alpha_a P_a |\psi\rangle$$

Measurement of $S_j$ — called syndrome — indicates if the system is in the code space $C$.

Given syndrome $(m_1, \ldots m_{n-k}) \in \{-1, 1\}^{n-k}$, we know that an error $E$ occurred with $ES_j = m_j S_j E$ since

$$S_j E |\psi\rangle = \pm ES_j |\psi\rangle = \pm E |\psi\rangle.$$
We only consider errors in $\mathcal{P}_n$ since they form a basis for all matrix: for $E = \sum_a \alpha_a P_a$

$$E|\psi\rangle = \sum_a \alpha_a P_a|\psi\rangle$$

Measurement of $S_j$ — called syndrome — indicates if the system is in the code space $C$.

Given syndrome $(m_1, \ldots m_{n-k}) \in \{-1, 1\}^{n-k}$, we know that an error $E$ occurred with $ES_j = m_j S_j E$ since

$$S_j E|\psi\rangle = \pm ES_j|\psi\rangle = \pm E|\psi\rangle.$$
Quantum error correction

Stabilizer formalism

Error syndrome

Example

The code specified by \( S = \langle ZZ11, 11ZZ \rangle \) corrects single-bit flip errors:

\[
\begin{align*}
1111 & \Rightarrow (1, 1) \\
X111 & \Rightarrow (-1, 1) \\
11X11 & \Rightarrow (-1, -1) \\
111X & \Rightarrow (1, -1).
\end{align*}
\]

- In the above example, the possible errors are in one-to-one mapping with the syndromes, so can obviously be corrected by re-applying the identified error.
- The QEC condition says that errors are correctable if they can be identified up to a stabilizer transformation.
  - Errors \( E \) and \( ES \) for \( S \in S \) have the same syndrome, so cannot be distinguished.
  - But the same correction procedure works for both errors: \( EES|\psi\rangle = S|\psi\rangle = |\psi\rangle \).
- The code \( S \) corrects \( \{E_a\} \) iff \( E_aE_b \not\in N(S) - S \) for all \( (a, b) \).
Error syndrome

Example

The code specified by \( S = \langle ZZ1l, 1lZZ \rangle \) corrects single-bit flip errors:

- \( 1l1l1l \Rightarrow (1, 1) \)
- \( X1l1l \Rightarrow (-1, 1) \)
- \( 1lX1l \Rightarrow (-1, -1) \)
- \( 1l1lX \Rightarrow (1, -1) \).

• In the above example, the possible errors are in one-to-one mapping with the syndromes, so can obviously be corrected by re-applying the identified error.

• The QEC condition says that errors are correctable if they can be identified up to a stabilizer transformation.
  
  Errors \( E \) and \( ES \) for \( S \in S \) have the same syndrome, so cannot be distinguished.
  
  But the same correction procedure works for both errors: \( EES|\psi\rangle = S|\psi\rangle = |\psi\rangle \).

• The code \( S \) corrects \( \{E_a\} \) iff \( E_aE_b \notin N(S) - S \) for all \( (a, b) \).
Error syndrome

Example

The code specified by \( S = \langle ZZ \mathbb{1}, \mathbb{1}ZZ \rangle \) corrects single-bit flip errors:

- \( 1\mathbb{1}1\mathbb{1} \Rightarrow (1, 1) \)
- \( X1\mathbb{1}1 \Rightarrow (-1, 1) \)
- \( 1\mathbb{1}X1\mathbb{1} \Rightarrow (-1, -1) \)
- \( 1\mathbb{1}1\mathbb{1}X \Rightarrow (1, -1) \).

In the above example, the possible errors are in one-to-one mapping with the syndromes, so can obviously be corrected by re-applying the identified error.

The QEC condition says that errors are correctable if they can be identified up to a stabilizer transformation.

- Errors \( E \) and \( ES \) for \( S \in S \) have the same syndrome, so cannot be distinguished.
- But the same correction procedure works for both errors: \( EES|\psi\rangle = S|\psi\rangle = |\psi\rangle \).

The code \( S \) corrects \( \{E_a\} \) iff \( E_aE_b \notin N(S) - S \) for all \( (a, b) \).
In the above example, the possible errors are in one-to-one mapping with the syndromes, so can obviously be corrected by re-applying the identified error.

The QEC condition says that errors are correctable if they can be identified up to a stabilizer transformation.

- Errors $E$ and $ES$ for $S \in S$ have the same syndrome, so cannot be distinguished.
- But the same correction procedure works for both errors: $EES|\psi\rangle = S|\psi\rangle = |\psi\rangle$.

The code $S$ corrects $\{E_a\}$ iff $E_aE_b \notin N(S) - S$ for all $(a, b)$. 
We have a code subspace $C$, now we need to decide how to embed the $k$ logical qubits in it.

Encoded operation should map code states to code states, so be in $N(S)$.

Logical Pauli operators $\overline{X}_i, \overline{Z}_i \in \mathcal{P}_n, i = 1, \ldots, k$:
- Same commutation relations as $X_i$ and $Z_i$.
- Map code states to code states: $[S_j, \overline{X}_i] = [S_j, \overline{Z}_j] = 0$. 
Encoding

- We have a code subspace \( C \), now we need to decide how to embed the \( k \) logical qubits in it.
- Encoded operation should map code states to code states, so be in \( N(S) \).
- Logical Pauli operators \( \overline{X}_i, \overline{Z}_i \in \mathcal{P}_n, i = 1, \ldots, k \):
  - Same commutation relations as \( X_i \) and \( Z_i \).
  - Map code states to code states: \( [S_j, \overline{X}_i] = [S_j, \overline{Z}_j] = 0 \).
We have a code subspace $C$, now we need to decide how to imbed the $k$ logical qubits in it.

Encoded operation should map code states to code states, so be in $N(S)$.

Logical Pauli operators $\overline{X}_i, \overline{Z}_i \in \mathcal{P}_n, i = 1, \ldots, k$:
- Same commutation relations as $X_i$ and $Z_i$.
- Map code states to code states: $[S_j, \overline{X}_i] = [S_j, \overline{Z}_j] = 0$. 
Encoding

- Can specify code and encoding by a Clifford transformation $U$:

\[
\begin{align*}
X_1 & \iff \overline{X}_1 \\
\vdots & \\
X_k & \iff \overline{X}_k \\
X_{k+1} & \iff T_1 \\
\vdots & \\
X_n & \iff T_{n-k} \\
Z_1 & \iff \overline{Z}_1 \\
\vdots & \\
Z_k & \iff \overline{Z}_k \\
Z_{k+1} & \iff S_1 \\
\vdots & \\
Z_n & \iff S_{n-k}
\end{align*}
\]
### Shor’s code

**Example**

<table>
<thead>
<tr>
<th>$S_1$</th>
<th>$ZZ11$</th>
<th>$11111$</th>
<th>$11111$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_2$</td>
<td>$11ZZ$</td>
<td>$11111$</td>
<td>$11111$</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$11111$</td>
<td>$ZZ11$</td>
<td>$11111$</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$11111$</td>
<td>$11ZZ$</td>
<td>$11111$</td>
</tr>
<tr>
<td>$S_5$</td>
<td>$11111$</td>
<td>$11111$</td>
<td>$ZZ11$</td>
</tr>
<tr>
<td>$S_6$</td>
<td>$11111$</td>
<td>$11111$</td>
<td>$11ZZ$</td>
</tr>
<tr>
<td>$S_7$</td>
<td>$XXX$</td>
<td>$XXX$</td>
<td>$11111$</td>
</tr>
<tr>
<td>$S_8$</td>
<td>$11111$</td>
<td>$XXX$</td>
<td>$XXX$</td>
</tr>
</tbody>
</table>

\[ \overline{X} = XXX \quad XXX \quad XXX \]
\[ \overline{Z} = ZZZ \quad ZZZ \quad ZZZ \]

- Encodes 1 logical qubit into 9 qubits.
- Protects against any single-qubit error.
Outline

1. Introduction
   - Context
   - Results summary

2. Quantum error correction
   - Definitions
   - Stabilizer formalism

3. Noiseless subsystems
   - Definition
   - Generalization

4. Operator quantum error correction
   - Definition
   - Conditions
   - Stabilizer formalism

5. Summary
Given map $\mathcal{E} = \{E_a\}$, consider $\mathcal{A} = \text{Alg}\{E_a, E_a^\dagger\}$.

It decomposes as

$$\mathcal{A} \cong \bigoplus_J 1_{d_J} \otimes M_{n_J}$$

Consider one $J$ factor (the largest $d_J$) and write $H = (A \otimes B) \oplus C^\perp$.

Any matrix of the form $\rho^A \otimes 1_B$ is a fixe point of $\mathcal{E}$.

The $A$ sector is called a noiseless subsystem of $\mathcal{E}$.

In the special case where $\text{dim}(B) = 1$, $A$ is called a decoherence free subspace.
Standard definition

- Given map $\mathcal{E} = \{E_a\}$, consider $\mathcal{A} = \text{Alg}\{E_a, E_a^\dagger\}$.
- It decomposes as
  \[
  \mathcal{A} \simeq \bigoplus_J 1_{d_J} \otimes M_{n_J}
  \]
- Consider one $J$ factor (the largest $d_J$) and write $H = (A \otimes B) \oplus C^\perp$.
- Any matrix of the form $\rho^A \otimes 1^B$ is a fixe point of $\mathcal{E}$.
- The $A$ sector is called a noiseless subsystem of $\mathcal{E}$.
- In the special case where $\text{dim}(B) = 1$, $A$ is called a decoherence free subspace.
Given map $\mathcal{E} = \{E_a\}$, consider $\mathcal{A} = \text{Alg}\{E_a, E_a^\dagger\}$.

It decomposes as

$$\mathcal{A} \simeq \bigoplus_{J} 1_{d_J} \otimes M_{n_J}$$

Consider one $J$ factor (the largest $d_J$) and write $H = (A \otimes B) \oplus C^\perp$.

Any matrix of the form $\rho^A \otimes 1_{d_B}$ is a fixed point of $\mathcal{E}$.

The $A$ sector is called a noiseless subsystem of $\mathcal{E}$.

In the special case where $\text{dim}(B) = 1$, $A$ is called a decoherence free subspace.
Standard definition

- Given map $\mathcal{E} = \{E_a\}$, consider $\mathcal{A} = \text{Alg}\{E_a, E_a^\dagger\}$.
- It decomposes as
  $$\mathcal{A} \simeq \bigoplus_J 1_{d_J} \otimes M_{n_J}$$
- Consider one $J$ factor (the largest $d_J$) and write $H = (A \otimes B) \oplus C^\perp$.
- Any matrix of the form $\rho^A \otimes 1_B$ is a fixe point of $\mathcal{E}$.
- The $A$ sector is called a noiseless subsystem of $\mathcal{E}$.
- In the special case where $\text{dim}(B) = 1$, $A$ is called a decoherence free subspace.
Given map $\mathcal{E} = \{E_a\}$, consider $\mathcal{A} = \text{Alg}\{E_a, E_a^\dagger\}$.

It decomposes as

$$\mathcal{A} \simeq \bigoplus_J 1_{d_J} \otimes M_{n_J}$$

Consider one $J$ factor (the largest $d_J$) and write $H = (A \otimes B) \oplus C^\perp$.

Any matrix of the form $\rho^A \otimes 1^B$ is a fixe point of $\mathcal{E}$.

The $A$ sector is called a noiseless subsystem of $\mathcal{E}$.

In the special case where $\text{dim}(B) = 1$, $A$ is called a decoherence free subspace.
Given map $\mathcal{E} = \{E_a\}$, consider $\mathcal{A} = \text{Alg}\{E_a, E_a^\dagger\}$.

It decomposes as

$$\mathcal{A} \simeq \bigoplus_J 1_{d_J} \otimes M_{n_J}$$

Consider one $J$ factor (the largest $d_J$) and write $H = (A \otimes B) \oplus C^\perp$.

Any matrix of the form $\rho^A \otimes 1^B$ is a fixe point of $\mathcal{E}$.

The $A$ sector is called a noiseless subsystem of $\mathcal{E}$.

In the special case where $\text{dim}(B) = 1$, $A$ is called a decoherence free subspace.
Collective rotation

Example (For physicists...)

- Consider applying a random collective rotation to 3 qubits:
  \[ \mathcal{E}(\rho) = \int_{SU(2)} U^{\otimes 3} \rho(U^{\dagger})^{\otimes 3} dU \]

- \( \mathcal{A} \) is generated by \( \{ J_x, J_y, J_z \} \) where \( J_x = X^{1111} + 11X^{11} + 111X \), etc.
- From addition of angular momentum, we have \( \mathcal{A} = \mathbb{C}^2 \otimes \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_{\frac{3}{2}} \).
- The noiseless algebra is generated by permutations of the qubits, which obviously commute with the noise operators.
Example (For physicists...)

- Consider applying a random collective rotation to 3 qubits:

\[ \mathcal{E}(\rho) = \int_{SU(2)} U^{\otimes 3} \rho (U^\dagger)^{\otimes 3} dU \]

- \( A \) is generated by \( \{ J_x, J_y, J_z \} \) where \( J_x = X111 + 11X11 + 111X \), etc.

- From addition of angular momentum, we have \( A = \mathbb{C}^2 \otimes \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_{\frac{3}{2}} \).

- The noiseless algebra is generated by permutations of the qubits, which obviously commute with the noise operators.
Example (For physicists...)

- Consider applying a random collective rotation to 3 qubits:
  \[ \mathcal{E}(\rho) = \int_{SU(2)} U \otimes^3 \rho (U^\dagger) \otimes^3 dU \]

- \( \mathcal{A} \) is generated by \( \{J_x, J_y, J_z\} \) where
  \[ J_x = X11l + 11X1 + 111X, \text{ etc.} \]

- From addition of angular momentum, we have
  \[ \mathcal{A} = \mathbb{C}^2 \otimes \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_{\frac{3}{2}}. \]

- The noiseless algebra is generated by permutations of the qubits,
  which obviously commute with the noise operators.
Collective rotation

Example (For physicists...)

Consider applying a random collective rotation to 3 qubits:

$$\mathcal{E}(\rho) = \int_{SU(2)} U^{\otimes 3} \rho (U^\dagger)^{\otimes 3} dU$$

- $\mathcal{A}$ is generated by $\{J_x, J_y, J_z\}$ where $J_x = X1\!\!1\!\!1 + 1\!\!1X\!\!1\!\!1 + 1\!\!11\!\!X$, etc.
- From addition of angular momentum, we have $\mathcal{A} = \mathbb{C}^2 \otimes \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_{\frac{3}{2}}$.
- The noiseless algebra is generated by permutations of the qubits, which obviously commute with the noise operators.
Outline

1. Introduction
   - Context
   - Results summary

2. Quantum error correction
   - Definitions
   - Stabilizer formalism

3. Noiseless subsystems
   - Definition
   - Generalization

4. Operator quantum error correction
   - Definition
   - Conditions
   - Stabilizer formalism

5. Summary
We don’t care about the fate of the $B$ subsystem!

**Definition (Noiseless subsystem, KLP05)**

Given an error map $\mathcal{E}$ and a decomposition $H = C \oplus C^\perp = (A \otimes B) \oplus C^\perp$, the subsystem $A$ is called **noiseless** if for all $\rho^A$ and $\rho^B$, there exists a $\rho'^B$ such that

$$\mathcal{E}(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B.$$ 

It is enough to check the condition for $\rho^B = 1_I$.

**Theorem (KLP05)**

The subsystem $A$ is noiseless for error map $\mathcal{E} = \{E_a\}$ if and only if

$$E_a P_C = 1_I^A \otimes g^B_a \quad \forall \ a.$$
Generalized definition

- We don’t care about the fate of the $B$ subsystem!

**Definition (Noiseless subsystem, KLP05)**

Given an error map $\mathcal{E}$ and a decomposition $H = C \oplus C^\perp = (A \otimes B) \oplus C^\perp$, the subsystem $A$ is called **noiseless** if for all $\rho^A$ and $\rho^B$, there exists a $\rho'^B$ such that

$$\mathcal{E}(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B.$$ 

- It is enough to check the condition for $\rho^B = 1_l$.

**Theorem (KLP05)**

The subsystem $A$ is noiseless for error map $\mathcal{E} = \{E_a\}$ if and only if

$$E_a P_C = 1_l^A \otimes g_a^B \ \forall \ a.$$
We don’t care about the fate of the $B$ subsystem!

**Definition (Noiseless subsystem, KLP05)**

Given an error map $\mathcal{E}$ and a decomposition $H = C \oplus C^\perp = (A \otimes B) \oplus C^\perp$, the subsystem $A$ is called **noiseless** if for all $\rho^A$ and $\rho^B$, there exists a $\rho'^B$ such that

$$\mathcal{E}(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B.$$ 

It is enough to check the condition for $\rho^B = 1_l$.

**Theorem (KLP05)**

The subsystem $A$ is noiseless for error map $\mathcal{E} = \{E_a\}$ if and only if

$$E_a P_C = 1_l^A \otimes g_a^B \ \forall \ a.$$
Generalized definition

- We don’t care about the fate of the $B$ subsystem!

**Definition (Noiseless subsystem, KLP05)**

Given an error map $\mathcal{E}$ and a decomposition $H = C \oplus C^\perp = (A \otimes B) \oplus C^\perp$, the subsystem $A$ is called **noiseless** if for all $\rho^A$ and $\rho^B$, there exists a $\rho'^B$ such that

$$\mathcal{E}(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B.$$  

- It is enough to check the condition for $\rho^B = 1_I$.

**Theorem (KLP05)**

The subsystem $A$ is noiseless for error map $\mathcal{E} = \{E_a\}$ if and only if

$$E_a P_C = 1_I^A \otimes g^B_a \ \forall \ a.$$
Definition (Correctable, KLP05)

Given an error map $\mathcal{E}$ and a decomposition $H = C \oplus C^\perp = (A \otimes B) \oplus C^\perp$, the subsystem $A$ is correctable if there exists a recovery map $\mathcal{R}$ such that for all $\rho^A$ and $\rho^B$, there exists a $\rho^{B'}$ such that

$$(\mathcal{R} \circ \mathcal{E})(\rho^A \otimes \rho^B) = \rho^A \otimes \rho^{B'}.$$ 

- In the special case where $\dim(B) = 1$, we recover the standard definition of correctability.
- In the special case where $\mathcal{R} = id$, we recover the (generalized) definition of a noiseless subsystem.
- In the doubly special case, we recover the definition of a decoherence free subspace.
Definition (Correctable, KLP05)

Given an error map \( \mathcal{E} \) and a decomposition \( H = C \oplus C^\perp \)
\( = (A \otimes B) \oplus C^\perp \), the subsystem \( A \) is correctable if there exists a
recovery map \( \mathcal{R} \) such that for all \( \rho^A \) and \( \rho^B \), there exists a \( \rho'^B \) such that
\[
(\mathcal{R} \circ \mathcal{E})(\rho^A \otimes \rho^B) = \rho^A \otimes \rho'^B.
\]

- In the special case where \( \dim(B) = 1 \), we recover the standard
definition of correctability.
- In the special case where \( \mathcal{R} = id \), we recover the (generalized)
definition of a noiseless subsystem.
- In the doubly special case, we recover the definition of a
decoherence free subspace.
Outline

1. Introduction
   - Context
   - Results summary

2. Quantum error correction
   - Definitions
   - Stabilizer formalism

3. Noiseless subsystems
   - Definition
   - Generalization

4. Operator quantum error correction
   - Definition
   - Conditions
   - Stabilizer formalism

5. Summary
The subsystem $A$ is correctable for error map $\mathcal{E} = \{E_a\}$ if and only if

$$PC E_a^\dagger E_b PC = 1^A \otimes g_{ab}^B$$

\forall a, b.

This is a non-Abelian version of the standard error correction condition, where $g_{ab}$ is a scalar.

Since all other known error corrections techniques are a special case of this:

- Standard QECC: $B$ is one-dimensional.
- Noiseless subsystem: recovery is trivial.
- Decoherence-free subspace: $B$ is one-dimensional and recovery is trivial.

we have a universal condition.

Necessity, KLP05.

Quite similar to the one used in standard QEC.
Theorem (KLP05,NP05)

The subsystem $A$ is correctable for error map $\mathcal{E} = \{E_a\}$ if and only if

$$P_CE_a^+E_bP_C = 1^{A} \otimes g_{ab}^{B} \forall a, b.$$  

- This is a non-Abelian version of the standard error correction condition, where $g_{ab}$ is a scalar.
- Since all other known error corrections techniques are a special case of this
  - Standard QECC: $B$ is one-dimensional.
  - Noiseless subsystem: recovery is trivial.
  - Decoherence-free subspace: $B$ is one-dimensional and recovery is trivial.

we have a universal condition.

Necessity, KLP05.

Quite similar to the one used in standard QEC.
The subsystem $A$ is correctable for error map $\mathcal{E} = \{ E_a \}$ if and only if

$$P_C E_a^\dagger E_b P_C = 11^A \otimes g_{ab}^B \quad \forall \ a, b.$$ 

- This is a non-Abelian version of the standard error correction condition, where $g_{ab}$ is a scalar.
- Since all other known error corrections techniques are a special case of this
  - Standard QECC: $B$ is one-dimensional.
  - Noiseless subsystem: recovery is trivial.
  - Decoherence-free subspace: $B$ is one-dimensional and recovery is trivial.
- we have a universal condition.

Necessity, KLP05.

Quite similar to the one used in standard QEC.
Condition

Theorem (KLP05,NP05)

The subsystem $A$ is correctable for error map $\mathcal{E} = \{E_a\}$ if and only if

$$P_C E_a^\dagger E_b P_C = 11_A \otimes g_{ab}^B \quad \forall \ a, b.$$ 

- This is a non-Abelian version of the standard error correction condition, where $g_{ab}$ is a scalar.
- Since all other known error corrections techniques are a special case of this
  - Standard QECC: $B$ is one-dimensional.
  - Noiseless subsystem: recovery is trivial.
  - Decoherence-free subspace: $B$ is one-dimensional and recovery is trivial.

we have a universal condition.

Necessity, KLP05.

Quite similar to the one used in standard QEC.
Condition

Sufficiency, NP05.

- Information theoretic
  - Generalizes coherent information and data processing inequality.
- Introduce reference systems $R_A$ and $R_B$ that are copies of $A$ and $B$.
- Define the states $|\alpha\rangle = \sum_j |j\rangle_A |j\rangle_{R_A}$ and $|\beta\rangle = \sum_k |k\rangle_B |k\rangle_{R_B}$.
- Purify the map $\mathcal{E}$ into $U$ with an environment $E$:

$$U |\psi\rangle_{AB} |0\rangle_E = \sum_a (E_a \otimes 1_1) |\psi\rangle_{AB} |a\rangle_E$$

- Apply the map $U \otimes 1_{R_A R_B}$ to the state $|\psi\rangle = |\alpha\rangle_{AR_A} |\beta\rangle_{BR_B} |0\rangle_E$ to obtain $|\psi'\rangle = U |\psi\rangle$.
- The condition becomes $S(\rho_A) = S(\rho'_H) - S(\rho'_{RB E})$. 
### Sufficiency, NP05.

- **Information theoretic**
  - Generalizes coherent information and data processing inequality.
- Introduce reference systems $R_A$ and $R_B$ that are copies of $A$ and $B$.
- Define the states $|\alpha\rangle = \sum_j |j\rangle_A |j\rangle_{R_A}$ and $|\beta\rangle = \sum_k |k\rangle_B |k\rangle_{R_B}$.
- Purify the map $\mathcal{E}$ into $U$ with an environment $E$:
  \[
  U|\psi\rangle_{AB}|0\rangle_E = \sum_a (E_a \otimes 1)|\psi\rangle_{AB}|a\rangle_E
  \]
- Apply the map $U \otimes 11_{R_AR_B}$ to the state $|\psi\rangle = |\alpha\rangle_{AR_A}|\beta\rangle_{BR_B}|0\rangle_E$ to obtain $|\psi'\rangle = U|\psi\rangle$.
- The condition becomes $S(\rho_A) = S(\rho'_H) - S(\rho'_{RBE})$. 

---

**Condition**
Sufficiency, NP05.

- Information theoretic
  - Generalizes coherent information and data processing inequality.
- Introduce reference systems $R_A$ and $R_B$ that are copies of $A$ and $B$.
- Define the states $|\alpha\rangle = \sum_j |j\rangle_A |j\rangle_{RA}$ and $|\beta\rangle = \sum_k |k\rangle_B |k\rangle_{RB}$.
- Purify the map $\mathcal{E}$ into $U$ with an environment $E$:
  \[
  U|\psi\rangle_{AB}|0\rangle_E = \sum_a (E_a \otimes 11)|\psi\rangle_{AB}|a\rangle_E
  \]
- Apply the map $U \otimes 11_{R_A R_B}$ to the state $|\psi\rangle = |\alpha\rangle_{AR_A}|\beta\rangle_{BR_B}|0\rangle_E$ to obtain $|\psi'\rangle = U|\psi\rangle$.
- The condition becomes $S(\rho_A) = S(\rho'_{H}) - S(\rho'_{R_B E})$. 
**Condition**

**Sufficiency, NP05.**

- Information theoretic
  - Generalizes coherent information and data processing inequality.
- Introduce reference systems $R_A$ and $R_B$ that are copies of $A$ and $B$.
- Define the states $|\alpha\rangle = \sum_j |j\rangle_A |j\rangle_R$ and $|\beta\rangle = \sum_k |k\rangle_B |k\rangle_R$.
- Purify the map $\mathcal{E}$ into $U$ with an environment $E$:

\[
U|\psi\rangle_{AB}|0\rangle_E = \sum_a (E_a \otimes 1_{R_A R_B})|\psi\rangle_{AB}|a\rangle_E
\]

- Apply the map $U \otimes 1_{R_A R_B}$ to the state $|\psi\rangle = |\alpha\rangle_{AR_A} |\beta\rangle_{BR_B}|0\rangle_E$ to obtain $|\psi'\rangle = U|\psi\rangle$.
- The condition becomes $S(\rho_A) = S(\rho'_{H}) - S(\rho'_{RB E})$. 
Operator quantum error correction

**Condition**

**Sufficiency, NP05.**

- Information theoretic
  - Generalizes coherent information and data processing inequality.
- Introduce reference systems $R_A$ and $R_B$ that are copies of $A$ and $B$.
- Define the states $|\alpha\rangle = \sum_j |j\rangle_A |j\rangle_{RA}$ and $|\beta\rangle = \sum_k |k\rangle_B |k\rangle_{RB}$.
- Purify the map $E$ into $U$ with an environment $E$:
  \[
  U|\psi\rangle_{AB}|0\rangle_E = \sum_a (E_a \otimes 1)|\psi\rangle_{AB}|a\rangle_E
  \]
- Apply the map $U \otimes 1_{RaRB}$ to the state $|\psi\rangle = |\alpha\rangle_{AR_A} |\beta\rangle_{BR_B} |0\rangle_E$ to obtain $|\psi'\rangle = U|\psi\rangle$.
- The condition becomes $S(\rho_A) = S(\rho'_H) - S(\rho'_{RB,E})$. 
**Condition**

**Sufficiency, NP05.**

- Information theoretic
  - Generalizes coherent information and data processing inequality.
- Introduce reference systems $R_A$ and $R_B$ that are copies of $A$ and $B$.
- Define the states $|\alpha\rangle = \sum_j |j\rangle_A |j\rangle_{RA}$ and $|\beta\rangle = \sum_k |k\rangle_B |k\rangle_{RB}$.
- Purify the map $\mathcal{E}$ into $U$ with an environment $E$:
  \[ U|\psi\rangle_{AB}|0\rangle_E = \sum_a (E_a \otimes 1_1)|\psi\rangle_{AB}|a\rangle_E \]
- Apply the map $U \otimes 1_1_{R_AR_B}$ to the state $|\psi\rangle = |\alpha\rangle_{RA}|\beta\rangle_{BR_B}|0\rangle_E$ to obtain $|\psi'\rangle = U|\psi\rangle$.
- The condition becomes $S(\rho_A) = S(\rho'_H) - S(\rho'_{RB_E})$. 

---

David Poulin (Queensland)  Operator Quantum Error Correction  Montréal 2006  30 / 38
Subaditivity implies $S(\rho'_R\rho E) \leq S(\rho'_R) + S(\rho'_E)$, with equality if and only if $\rho'_R\rho E = \rho'_R \otimes \rho'_E$.

Assuming the above holds, we write $|\psi'\rangle$ in the Schmidt form for the partition $H : R_A R_B E$:

$$|\psi'\rangle = \sum_{jk} \lambda_k |\phi_{jk}\rangle_H \otimes |j\rangle_R \otimes |k\rangle_{BE}.$$ 

We define the projectors $P_k$ on $H$ by $P_k |\phi_{jk}\rangle_H = \delta_{kk'} |\phi_{jk}\rangle_H$.

We perform the measurement $\{P_k\}$ and get some outcome $k$.

Conditioned on this result, we apply the transformation $U_k : |\phi_{jk}\rangle_H \rightarrow |j\rangle_A \otimes |0\rangle_B$.

The resulting state is $|\alpha\rangle_{RA} \otimes |0\rangle_B \otimes |k\rangle_{BE}$. 

---

David Poulin (Queensland)  Operator Quantum Error Correction  Montréal 2006  31 / 38
Subaditivity implies $S(\rho'_{RA_{RB}E}) \leq S(\rho'_{RA}) + S(\rho'_{RB}E)$, with equality if and only if $\rho'_{RA_{RB}E} = \rho'_{RA} \otimes \rho'_{RB}E$.

Assuming the above holds, we write $|\psi'\rangle$ in the Schmidt form for the partition $H : RA_{RB}E$:

$$|\psi'\rangle = \sum_{jk} \lambda_k |\phi_{jk}\rangle_H \otimes |j\rangle_{RA} \otimes |k\rangle_{RB}E.$$ 

We define the projectors $P_k$ on $H$ by $P_k |\phi_{jk}\rangle_H = \delta_{kk'} |\phi_{jk}\rangle_H$.

We perform the measurement $\{P_k\}$ and get some outcome $k$.

Conditioned on this result, we apply the transformation $U_k : |\phi_{jk}\rangle_H \mapsto |j\rangle_A \otimes |0\rangle_B$.

The resulting state is $|\alpha\rangle_{AR} \otimes |0\rangle_B \otimes |k\rangle_{RB}E$. 


Subaditvity implies $S(\rho'_{RA}^{RBE}) \leq S(\rho'_{RA}) + S(\rho'_{RBE})$, with equality if and only if $\rho'_{RA}^{RBE} = \rho'_{RA} \otimes \rho'_{RBE}$.

Assuming the above holds, we write $|\psi'\rangle$ in the Schmidt form for the partition $H : RA : RB : E$:

$$|\psi'\rangle = \sum_{jk} \lambda_k |\phi_{jk}\rangle_H \otimes |j\rangle_{RA} \otimes |k\rangle_{RBE}.$$ 

We define the projectors $P_k$ on $H$ by $P_k |\phi_{jk}\rangle_H = \delta_{kk'} |\phi_{jk}\rangle_H$.

We perform the measurement $\{P_k\}$ and get some outcome $k$.

Conditioned on this result, we apply the transformation $U_k : |\phi_{jk}\rangle_H \mapsto |j\rangle_A \otimes |0\rangle_B$.

The resulting state is $|\alpha\rangle_{AR} \otimes |0\rangle_B \otimes |k\rangle_{RBE}$. 
Subaditivity implies $S(\rho'_{RA_{RB}E}) \leq S(\rho'_{RA}) + S(\rho'_{RB}E)$, with equality if and only if $\rho'_{RA_{RB}E} = \rho'_{RA} \otimes \rho'_{RB}E$.

Assuming the above holds, we write $|\psi'\rangle$ in the Schmidt form for the partition $H : R_AR_B E$:

$$|\psi'\rangle = \sum_{jk} \lambda_{jk} |\phi_{jk}\rangle_H \otimes |j\rangle_{RA} \otimes |k\rangle_{RB}E.$$ 

We define the projectors $P_k$ on $H$ by $P_k |\phi_{jk'}\rangle_H = \delta_{kk'} |\phi_{jk}\rangle_H$.

We perform the measurement $\{P_k\}$ and get some outcome $k$.

Conditioned on this result, we apply the transformation $U_k : |\phi_{jk}\rangle_H \mapsto |j\rangle_A \otimes |0\rangle_B$.

The resulting state is $|\alpha\rangle_{AR_A} \otimes |0\rangle_B \otimes |k\rangle_{RB}E$. 
Subaditivity implies \( S(\rho'_{R_A R_B E}) \leq S(\rho'_{R_A}) + S(\rho'_{R_B E}) \), with equality if and only if \( \rho'_{R_A R_B E} = \rho'_{R_A} \otimes \rho'_{R_B E} \).

Assuming the above holds, we write \( |\psi'\rangle \) in the Schmidt form for the partition \( H : R_A R_B E : \)

\[
|\psi'\rangle = \sum_{jk} \lambda_k |\phi_{jk}\rangle_H \otimes |j\rangle_{R_A} \otimes |k\rangle_{R_B E}.
\]

We define the projectors \( P_k \) on \( H \) by \( P_k |\phi_{jk'}\rangle_H = \delta_{kk'} |\phi_{jk}\rangle_H \).

We perform the measurement \( \{P_k\} \) and get some outcome \( k \).

Conditioned on this result, we apply the transformation \( U_k : |\phi_{jk}\rangle_H \mapsto |j\rangle_A \otimes |0\rangle_B \).

The resulting state is \( |\alpha\rangle_{A R_A} \otimes |0\rangle_B \otimes |k\rangle_{R_B E} \).
Subaditivitiy implies $S(\rho'_{RA}R_{BE}) \leq S(\rho'_{RA}) + S(\rho'_{RBE})$, with equality if and only if $\rho'_{RA}R_{BE} = \rho'_{RA} \otimes \rho'_{RBE}$.

Assuming the above holds, we write $|\psi'\rangle$ in the Schmidt form for the partition $H : RA R_{BE}$:

$$|\psi'\rangle = \sum_{jk} \lambda_k |\phi_{jk}\rangle_H \otimes |j\rangle_R A \otimes |k\rangle_{RBE}. $$

We define the projectors $P_k$ on $H$ by $P_k |\phi_{jk}\rangle_H = \delta_{kk'} |\phi_{jk}\rangle_H$.

We perform the measurement $\{P_k\}$ and get some outcome $k$.

Conditioned on this result, we apply the transformation $U_k : |\phi_{jk}\rangle_H \mapsto |j\rangle_R A \otimes |0\rangle_B$.

The resulting state is $|\alpha\rangle_{RA} \otimes |0\rangle_B \otimes |k\rangle_{RBE}$. 
Consider the conditional entropy of $R_A$ given $H$:

$$-S(R_A|H) = S(H) - S(R_AH).$$

It represents the amount of information initially in $A$ still stored in $H$.

This quantity is monotone:

$$-S(R_A|H) \geq S(R'_A|H') \geq S(R''_A|H'') \geq \ldots$$

The above condition for OQEC is equivalent to

$$-S(R_A|H) = -S(R'_A|H').$$
Consider the conditional entropy of $R_A$ given $H$: 

$$-S(R_A|H) = S(H) - S(R_AH).$$

It represents the amount of information initially in $A$ still stored in $H$.

This quantity is monotone:

$$-S(R_A|H) \geq S(R_A'|H') \geq S(R_A''|H'') \geq \ldots$$

The above condition for OQEC is equivalent to

$$-S(R_A|H) = -S(R_A'|H').$$
Consider the conditional entropy of $R_A$ given $H$:
$$-S(R_A|H) = S(H) - S(R_AH).$$
It represents the amount of information initially in $A$ still stored in $H$.
This quantity is monotone:
$$-S(R_A|H) \geq S(R'_A|H') \geq S(R''_A|H'') \geq \ldots$$
The above condition for OQEC is equivalent to
$$-S(R_A|H) = -S(R'_A|H').$$
Consider the conditional entropy of $R_A$ given $H$:

$$-S(R_A|H) = S(H) - S(R_AH).$$

It represents the amount of information initially in $A$ still stored in $H$.

This quantity is monotone:

$$-S(R_A|H) \geq S(R'_A|H') \geq S(R''_A|H'') \geq \ldots$$

The above condition for OQEC is equivalent to

$$-S(R_A|H) = -S(R'_A|H').$$
Outline

1. Introduction
   - Context
   - Results summary

2. Quantum error correction
   - Definitions
   - Stabilizer formalism

3. Noiseless subsystems
   - Definition
   - Generalization

4. Operator quantum error correction
   - Definition
   - Conditions
   - Stabilizer formalism

5. Summary
In OQEC, there is a freedom in the encoding of information, \( \rho^A \otimes \rho^B \) and \( \rho^A \otimes \rho'^B \) are regarded as equivalent.

We need to introduce a group of equivalence relations between operations.

We already have one, the stabilizer: encoded operators related by a stabilizer transformation \( U \) and \( US \) have the same effects on code states

\[
US|\psi\rangle = U|\psi\rangle
\]

But this group is Abelian, so is associated to a trivial factor of the Hilbert space.

We increase the size of this equivalence group by adding extra Pauli operators: \( \mathcal{G} = \langle S_1, \ldots, S_{n-k}, g_1^x, g_1^z, \ldots g_r^x, g_r^z \rangle \).

The QEC condition becomes \( E_a E_b \notin N(S) - \mathcal{G} \) for all \((a, b)\).
In OQEC, there is a freedom in the encoding of information, $\rho^A \otimes \rho^B$ and $\rho^A \otimes \rho'^B$ are regarded as equivalent.

We need to introduce a group of equivalence relations between operations.

We already have one, the stabilizer: encoded operators related by a stabilizer transformation $U$ and $US$ have the same effects on code states

$$US|\psi\rangle = U|\psi\rangle$$

But this group is Abelian, so is associated to a trivial factor of the Hilbert space.

We increase the size of this equivalence group by adding extra Pauli operators: $\mathcal{G} = \langle S_1, \ldots, S_{n-k}, g_1^x, g_1^z, \ldots, g_r^x, g_r^z \rangle$.

The QEC condition becomes $E_aE_b \notin N(S) - \mathcal{G}$ for all $(a, b)$. 

David Poulin (Queensland) 

Operator Quantum Error Correction 

Montréal 2006 

34 / 38
In OQEC, there is a freedom in the encoding of information, $\rho^A \otimes \rho^B$ and $\rho^A \otimes \rho'^B$ are regarded as equivalent.

We need to introduce a group of equivalence relations between operations.

We already have one, the stabilizer: encoded operators related by a stabilizer transformation $U$ and $US$ have the same effects on code states

$$US|\psi\rangle = U|\psi\rangle$$

But this group is Abelian, so is associated to a trivial factor of the Hilbert space.

We increase the size of this equivalence group by adding extra Pauli operators: $\mathcal{G} = \langle S_1, \ldots, S_{n-k}, g^x_1, g^z_1, \ldots g^x_r, g^z_r \rangle$.

The QEC condition becomes $E_aE_b \notin N(S) - \mathcal{G}$ for all $(a, b)$. 
Gauge qubits

- In OQEC, there is a freedom in the encoding of information, $\rho^A \otimes \rho^B$ and $\rho^A \otimes \rho'^B$ are regarded as equivalent.
- We need to introduce a group of equivalence relations between operations.
- We already have one, the stabilizer: encoded operators related by a stabilizer transformation $U$ and $US$ have the same effects on code states

$$US|\psi\rangle = U|\psi\rangle$$

- But this group is Abelian, so is associated to a trivial factor of the Hilbert space.
- We increase the size of this equivalence group by adding extra Pauli operators: $G = \langle S_1, \ldots, S_{n-k}, g_1^x, g_1^z, \ldots g_r^x, g_r^z \rangle$.
- The QEC condition becomes $E_aE_b \notin N(S) - G$ for all $(a, b)$. 
Gauge qubits

- In OQEC, there is a freedom in the encoding of information, $\rho^A \otimes \rho^B$ and $\rho^A \otimes \rho'^B$ are regarded as equivalent.

- We need to introduce a group of equivalence relations between operations.

- We already have one, the stabilizer: encoded operators related by a stabilizer transformation $U$ and $US$ have the same effects on code states

$$US|\psi\rangle = U|\psi\rangle$$

- But this group is Abelian, so is associated to a trivial factor of the Hilbert space.

- We increase the size of this equivalence group by adding extra Pauli operators: $\mathcal{G} = \langle S_1, \ldots, S_{n-k}, g^x_1, g^z_1, \ldots, g^x_r, g^z_r \rangle$.

- The QEC condition becomes $E_aE_b \notin N(S) - \mathcal{G}$ for all $(a, b)$.
In OQEC, there is a freedom in the encoding of information, \( \rho^A \otimes \rho^B \) and \( \rho^A \otimes \rho'^B \) are regarded as equivalent.

We need to introduce a group of equivalence relations between operations.

We already have one, the stabilizer: encoded operators related by a stabilizer transformation \( U \) and \( US \) have the same effects on code states

\[
US|\psi\rangle = U|\psi\rangle
\]

But this group is Abelian, so is associated to a trivial factor of the Hilbert space.

We increase the size of this equivalence group by adding extra Pauli operators: \( G = \langle S_1, \ldots, S_{n-k}, g^x_1, g^z_1, \ldots g^x_r, g^z_r \rangle \).

The QEC condition becomes \( E_a E_b \notin N(S) - G \) for all \( (a, b) \).
Operator quantum error correction

Stabilizer formalism

\[ U : \]
\[
\begin{align*}
X_1 & \leftrightarrow \overline{X}_1 \\
& \quad \quad \vdots \\
X_k & \leftrightarrow \overline{X}_k \\
X_{k+1} & \leftrightarrow g_1^x \\
& \quad \quad \vdots \\
X_{k+r} & \leftrightarrow g_r^x \\
X_{n-s} & \leftrightarrow T_1 \\
X_n & \leftrightarrow T_s \\
Z_1 & \leftrightarrow \overline{Z}_1 \\
& \quad \quad \vdots \\
Z_k & \leftrightarrow \overline{Z}_k \\
Z_{k+1} & \leftrightarrow g_1^z \\
& \quad \quad \vdots \\
Z_{k+r} & \leftrightarrow g_r^z \\
Z_{n-s} & \leftrightarrow S_1 \\
Z_n & \leftrightarrow S_s
\end{align*}
\]

David Poulin (Queensland)

Operator Quantum Error Correction

Montréal 2006 35 / 38
What’s the point of adding random qubits to the code?

- It is less constraint.
  - We replace stabilizer qubits by gauge qubits, so there are less syndromes to measure.
  - This is good for fault tolerance since syndrome measurements are an important source of errors.
  - We have more freedom in choosing encoded operators.

Example

- Collective flip channel \(\{111, XX\}\).
- We can use the code \(S = \{Z1\}\) with logical states \(|00\rangle\) and \(|01\rangle\).
- Instead, we can just encode in the parity, by defining the gauge group \(\langle XX, Z1, 11Z\rangle\).
- The scheme becomes passive.
What’s the point of adding random qubits to the code?

- It is less constraint.
  - We replace stabilizer qubits by gauge qubits, so there are less syndromes to measure.
  - This is good for fault tolerance since syndrome measurements are an important source of errors.
  - We have more freedom in choosing encoded operators.

Example

- Collective flip channel $\{1\bar{1}, XX\}$.
- We can use the code $S = \{Z1\bar{1}\}$ with logical states $|00\rangle$ and $|01\rangle$.
- Instead, we can just encode in the parity, by defining the gauge group $\langle XX, Z1\bar{1}, 1\bar{1}Z\rangle$.
- The scheme becomes passive.
Error correction

- What’s the point of adding random qubits to the code?
  - It is less constraint.
    - We replace stabilizer qubits by gauge qubits, so there are less syndromes to measure.
    - This is good for fault tolerance since syndrome measurements are an important source of errors.
    - We have more freedom in choosing encoded operators.

Example

- Collective flip channel \{1|1\}, XX\}.
- We can use the code \(S = \{Z1\}\) with logical states \(|00\rangle\) and \(|01\rangle\).
- Instead, we can just encode in the parity, by defining the gauge group \(\langle XX, Z1, 11Z\rangle\).
- The scheme becomes passive.
Shor’s code revisited

Example

- Shor’s code encodes $k = 1$ logical qubits into $n = 9$ physical qubits, thus requiring measurement of $n - k = 8$ stabilizers.
- It protects the information against any single-qubit error.
- We can replace 4 of the 8 stabilizer qubits by gauge qubits, without affecting the error correction capacities.

<table>
<thead>
<tr>
<th>$S_1$</th>
<th>$ZZ, 1, l$</th>
<th>$ZZ, 1, l$</th>
<th>$ZZ, 1, l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_2$</td>
<td>$1, l, ZZ$</td>
<td>$1, l, ZZ$</td>
<td>$1, l, ZZ$</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$XXX, , XXX$</td>
<td>$XXX, , XXX$</td>
<td>$1, l, 1, l, 1, l$</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$1, l, 1, l, 1, l$</td>
<td>$XXX, , XXX$</td>
<td>$XXX, , XXX$</td>
</tr>
</tbody>
</table>

- The overall state doesn’t remain pure, but the encoded information is unaffected.

$X = XXX\, \, XXX\, \, XXX$

$Z = ZZZ\, \, ZZZ\, \, ZZZ$
Summary

Conclusion

- OQEC builds on existing error correcting schemes and their generalization to provide a unified picture for protecting quantum information.
  - Universal necessary and sufficient condition for error correction.
  - Generalization of the notion of coherent information and data processing inequality.

- Open questions
  - How many gauge qubits can be added to a given code?
  - New information-theoretic notions useful for channel capacity?
  - Stabilizer formalism for mixed state measurement based QIP?
  - Generalize to approximate error correction.