Macroscopic observables

A formal description of NMR-type measurements

David Poulin

Institute for Quantum Computing
Perimeter Institute for Theoretical Physics
Macroscopic observables $A_N = \sum_{k=1}^{N} a^{(k)}$.

NMR measurement model.

Macroscopic observables on $N$ identically prepared quantum systems.

- Infinite limit $N = \infty$.
- State disturbance.
- Measurement accuracy-state disturbance tradeoff.

More NMR considerations.

- Room temperature.
- Conventional commutation argument.
- Is NMR QIP really quantum?
Avant propos

For sake of simplicity, we will restrict our attention to...
- Two-dimensional quantum systems.
- Pure states.
- Gaussian smoothing functions.

The general treatment can be found in quant-ph/0403212.
- Classical limit in closed quantum systems.
- Consistent histories.

The only extra ingredients are...
- Normalized Hamming weight $\Rightarrow$ Type of a string.
- Mixed states: church of the larger Hilbert space.
- Cauchy-Schwartz inequalities, ...
Macroscopic observables

General setting: A collection—sample—of $N$ quantum systems of the same nature—molecules—are measured jointly.
Macroscopic observables

General setting: A collection—sample—of $N$ quantum systems of the same nature—molecules—are measured jointly.

Let $a$ be an observable on a 2-dimensional molecule with eigenvalues/vectors

$$a|j\rangle = \lambda_j |j\rangle, \quad j = 0, 1.$$
Macroscopic observables

General setting: A collection—sample—of $N$ quantum systems of the same nature—molecules—are measured jointly.

- Let $a$ be an observable on a 2-dimensional molecule with eigenvalues/vectors

$$a|j\rangle = \lambda_j |j\rangle, \quad j = 0, 1.$$

- Consider the macroscopic observable $A_N = \sum_{k=1}^{N} a(k)$ where

$$a(k) = \underbrace{1 \otimes \ldots \otimes 1}_{k-1} \otimes a \otimes \underbrace{1 \otimes \ldots \otimes 1}_{N-k}$$
Macroscopic observables

General setting: A collection—sample—of \( N \) quantum systems of the same nature—molecules—are measured jointly.

Let \( a \) be an observable on a 2-dimensional molecule with eigenvalues/vectors

\[
a|j\rangle = \lambda_j |j\rangle, \quad j = 0, 1.
\]

Consider the macroscopic observable

\[
A_N = \sum_{k=1}^{N} a(k)
\]

where

\[
a(k) = 1 \otimes \ldots \otimes 1 \otimes a \otimes 1 \otimes \ldots \otimes 1
\]

\[
\quad k - 1 \quad \quad N - k
\]

The “computational basis" \( \{|X\rangle\}_{X=0\ldots2^{N-1}} \) where \( |X\rangle = |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle \) is defined with respect to observable \( a \).
Macroscopic observables

The states $|X\rangle$ are eigenstates of $A_N$. 

The eigenvalue of $|X\rangle$ depends only on the normalized Hamming weight of $X$: 

$$|X\rangle = \sum_{k=1}^{N} \lambda_{j_k} |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle \pm \left( \sum_{k=1}^{N} \lambda_{j_k} ight) |X\rangle \pm \left( \sum_{k=1}^{N} \lambda_{j_k} \right) |X\rangle \pm \ldots$$
Macroscopic observables

The states $|X\rangle$ are eigenstates of $A_N$.

$$A_N|X\rangle = A_N|j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle$$
The states $|X\rangle$ are eigenstates of $A_N$.

\[
A_N |X\rangle = A_N |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle \\
= \sum_{k=1}^{N} a_{(k)} |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle
\]
The states $|X\rangle$ are eigenstates of $A_N$.

$$A_N |X\rangle = A_N |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle$$

$$= \sum_{k=1}^{N} a_{(k)} |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle$$

$$= \sum_{k=1}^{N} \lambda_{j_k} |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle$$
The states $|X\rangle$ are eigenstates of $A_N$.

$$A_N |X\rangle = A_N |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle$$

$$= \sum_{k=1}^{N} a_{(k)} |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle$$

$$= \sum_{k=1}^{N} \lambda_{j_k} |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle$$

$$= \left[ \sum_{k=1}^{N} \lambda_{j_k} \right] |X\rangle$$
Macroscopic observables

The states $|X\rangle$ are eigenstates of $A_N$.

\[
A_N |X\rangle = A_N |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle = N \sum_{k=1}^{N} a_{(k)} |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle
\]

\[
= \sum_{k=1}^{N} \lambda_{jk} |j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle
\]

\[
= \left[ \sum_{k=1}^{N} \lambda_{jk} \right] |X\rangle = \left[ N (1 - |X|) \lambda_0 + N |X| \lambda_1 \right] |X\rangle.
\]

The eigenvalue of $|X\rangle$ depends only on the Normalized Hamming weight of $X$: $|X| = \frac{\# 1's \text{ in } X}{N}$.
Macroscopic observables

Therefore, the spectral decomposition of $A_N$ is

$$A_N = \sum_h \left[ N(1 - h)\lambda_0 + Nh\lambda_1 \right] \sum_{X:|X|=h} |X\rangle\langle X| = \sum_h A_h Q_h^{(N)}$$

where $h$ takes the values $0, \frac{1}{N}, \frac{2}{N}, \ldots 1$. 
Therefore, the spectral decomposition of $A_N$ is

$$A_N = \sum_h \left[ N(1 - h)\lambda_0 + Nh\lambda_1 \right] \sum_{X:|X|=h} |X\rangle\langle X| = \sum_h A_h Q_h^{(N)}$$

where $h$ takes the values $0, \frac{1}{N}, \frac{2}{N}, \ldots 1$.

Measuring observable $A_N$ corresponds to the von Neumann measurement composed of the Hamming projectors $Q_h^{(N)}$:

$$Q_h^{(N)} Q_{h'}^{(N)} = \delta_{hh'} Q_h^{(N)} \quad \text{and} \quad \sum_h Q_h^{(N)} = 1_l$$
Macroscopic observables

Therefore, the spectral decomposition of $A_N$ is

$$A_N = \sum_h \left[ N(1-h)\lambda_0 + Nh\lambda_1 \right] \sum_{X:|X|=h} |X\rangle \langle X| = \sum_h A_h Q_h^{(N)}$$

where $h$ takes the values $0, \frac{1}{N}, \frac{2}{N}, \ldots 1$.

- Measuring observable $A_N$ corresponds to the von Neumann measurement composed of the Hamming projectors $Q_h^{(N)}$:

$$Q_h^{(N)} Q_h^{(N)} = \delta_{h,h'} Q_h^{(N)} \quad \text{and} \quad \sum_h Q_h^{(N)} = 1$$

- Example: The total (bulk) magnetization of a sample of $N$ spin-$\frac{1}{2}$ particles tells us about the relative population of up and down states.
Macroscopic observables

What if our measurement of $A_N$ has finite accuracy?

$$\tilde{Q}_\ell^{(N)} = \sum_h \sqrt{q_h(\ell)} Q_h^{(N)}$$

where $q_h(\ell) \geq 0$ and $\sum_\ell q_h(\ell) = 1$.
Macroscopic observables

What if our measurement of $A_N$ has finite accuracy?

$$\tilde{Q}_\ell^{(N)} = \sum_h \sqrt{q_h(\ell)} Q_h^{(N)} \text{ where } q_h(\ell) \geq 0 \text{ and } \sum_{\ell} q_h(\ell) = 1$$

For example, $q_h(\ell) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ -\frac{(h-\ell)^2}{2\sigma^2} \right\}$. 
Macroscopic observables

What if our measurement of $A_N$ has finite accuracy?

$$\tilde{Q}_\ell^{(N)} = \sum_h \sqrt{q_h(\ell)} Q_h^{(N)}$$
where $q_h(\ell) \geq 0$ and $\sum_\ell q_h(\ell) = 1$

For example, $q_h(\ell) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(h-\ell)^2}{2\sigma^2} \right\}$.

$\sigma$ represents the accuracy of our measurement.
Macroscopic observables

What if our measurement of $A_N$ has finite accuracy?

$$\tilde{Q}_\ell^{(N)} = \sum_h \sqrt{q_h(\ell)} Q_h^{(N)}$$

where $q_h(\ell) \geq 0$ and $\sum_\ell q_h(\ell) = 1$

For example, $q_h(\ell) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ -\frac{(h-\ell)^2}{2\sigma^2} \right\}$.

$\sigma$ represents the accuracy of our measurement.

Since $\ell$ contains statistical fluctuations, it can in principle take any real value.
Macroscopic observables

What if our measurement of $A_N$ has finite accuracy?

$$\tilde{Q}_\ell^{(N)} = \sum_h \sqrt{q_h(\ell)} Q_h^{(N)}$$
where $q_h(\ell) \geq 0$ and $\sum_\ell q_h(\ell) = 1$

For example, $q_h(\ell) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ -\frac{(h-\ell)^2}{2\sigma^2} \right\}$.

$\sigma$ represents the accuracy of our measurement.

Since $\ell$ contains statistical fluctuations, it can in principle take any real value.

The $\tilde{Q}_\ell^{(N)}$ are the Kraus operators of a POVM

$$P(\tilde{Q}_\ell^{(N)} | \rho) = Tr\{\tilde{Q}_\ell^{(N)} \rho \tilde{Q}_\ell^{(N)} \} \quad \text{and} \quad \rho \xrightarrow{\ell} \rho|\ell = \frac{\tilde{Q}_\ell^{(N)} \rho \tilde{Q}_\ell^{(N)}}{P(\tilde{Q}_\ell^{(N)} | \rho)}$$
The sample is placed in a coiled wire.
The sample is placed in a coiled wire.

The current in the coil can be represented by a quantum field.

The qubit couples to a single mode of this field

\[ R_0 = \int r |r\rangle\langle r| \, dr. \]

One can think of \( R_0 \) as the observable corresponding to the amplitude of the current in the coil (more precisely the component of the current at the resonant frequency of the spin).

The coupling Hamiltonian is

\[ H_c = \gamma M_x \otimes P_0 \]

where:

1. \( M_x \) is the bulk magnetization in the \( x \) direction
   \[ M_x = \sum_{N_{k=1}} \sigma_x(k) = \sum_{h} \left( h - \frac{1}{2} \right) Q(N) h. \]

2. \( P_0 \) is the conjugate momentum of the field
   \[ [R_0, P_0] = i. \]
The sample is placed in a coiled wire.

- The current in the coil can be represented by a quantum field.
- The qubit couples to a single mode of this field

\[ R_0 = \int r |r\rangle\langle r|dr. \]

One can think of \( R_0 \) as the observable corresponding to the amplitude of the current in the coil (more precisely the component of the current at the resonant frequency of the spin).
The sample is placed in a coiled wire.

- The current in the coil can be represented by a quantum field.
- The qubit couples to a single mode of this field
  \[ R_0 = \int r |r\rangle\langle r| \, dr. \]
- One can think of \( R_0 \) as the observable corresponding to the amplitude of the current in the coil (more precisely the component of the current at the resonant frequency of the spin).
- The coupling Hamiltonian is \( H_c = \gamma M_x \otimes P_0 \) where:
The sample is placed in a coiled wire.

The current in the coil can be represented by a quantum field.

The qubit couples to a single mode of this field
\[ R_0 = \int r |r\rangle \langle r| dr. \]

One can think of \( R_0 \) as the observable corresponding to the amplitude of the current in the coil (more precisely the component of the current at the resonant frequency of the spin).

The coupling Hamiltonian is \( H_c = \gamma M_x \otimes P_0 \) where:

1. \( M_x \) is the bulk magnetization in the \( x \) direction

\[ M_x = \sum_{k=1}^{N} \sigma^x_{(k)} = \sum_{h} (h - \frac{1}{2}) Q_h^{(N)}. \]
The sample is placed in a coiled wire.

The current in the coil can be represented by a quantum field.

The qubit couples to a single mode of this field

$$R_0 = \int r |r\rangle\langle r| dr.$$

One can think of $R_0$ as the observable corresponding to the amplitude of the current in the coil (more precisely the component of the current at the resonant frequency of the spin).

The coupling Hamiltonian is $H_c = \gamma M_x \otimes P_0$ where:

1. $M_x$ is the bulk magnetization in the $x$ direction

$$M_x = \sum_{k=1}^{N} \sigma_x^{(k)} = \sum_{h}(h - \frac{1}{2})Q_h^{(N)}.$$

2. $P_0$ is the conjugate momentum of the field $[R_0, P_0] = i.$
After a time $t$, we measure the field in the coil and get a value $r$. For the molecule, this corresponds to performing the POVM $\tilde{Q}(N)$ with smoothing function $q_h(r) = |\phi(r - \gamma t N(h - 1/2))|^2$ where $\phi$ is the initial field configuration $|\phi\rangle = \int \phi(r)|r\rangle\,dr$. The measurement accuracy is determined by the initial spread of the field configuration $\sigma \approx \sqrt{\langle \phi | R_0^2 | \phi \rangle - \langle \phi | R_0 | \phi \rangle^2/N\gamma t}$. Actual measurement accuracy $\gg 1/\sqrt{N} \approx 10^{-10}$. 

QIP CIAR Meeting, Vancouver, May 2004 – p.9
After a time $t$, we measure the field in the coil and get a value $r$.

For the molecule, this corresponds to performing the POVM $\tilde{Q}_r^{(N)}$ with smoothing function

$$ q_h(r) = \left| \phi \left( r - \gamma t N (h - \frac{1}{2}) \right) \right|^2 $$

where $\phi$ is the initial field configuration $|\phi\rangle = \int \phi(r)|r\rangle dr$. 
After a time $t$, we measure the field in the coil and get a value $r$. For the molecule, this corresponds to performing the POVM $\tilde{Q}^{(N)}_r$ with smoothing function

$$q_h(r) = \left| \phi \left( r - \gamma t N \left( \hbar - \frac{1}{2} \right) \right) \right|^2$$

where $\phi$ is the initial field configuration $|\phi\rangle = \int \phi(r)|r\rangle dr$. The measurement accuracy is determined by the initial spread of the field configuration $\sigma \approx \sqrt{\langle \phi | R_0^2 | \phi \rangle - \langle \phi | R_0 | \phi \rangle^2} / N \gamma t$. 

QIP CIAR Meeting, Vancouver, May 2004 – p.9
After a time \( t \), we measure the field in the coil and get a value \( r \).

For the molecule, this corresponds to performing the POVM \( \tilde{Q}_r^{(N)} \) with smoothing function

\[
q_h(r) = \left| \phi \left( r - \gamma t N \left( h - \frac{1}{2} \right) \right) \right|^2
\]

where \( \phi \) is the initial field configuration \( |\phi\rangle = \int \phi(r)|r\rangle dr \).

The measurement accuracy is determined by the initial spread of the field configuration

\[
\sigma \approx \sqrt{\langle \phi | R_0^2 | \phi \rangle - \langle \phi | R_0 | \phi \rangle^2} / N \gamma t.
\]

Actual measurement accuracy \( \gg 1 / \sqrt{N} \approx 10^{-10} \).
This \textit{physical} description is equivalent to the following quantum circuit:

\[ |h\rangle \xrightarrow{R_0} \text{Shift} \xrightarrow{h} |h\rangle \]

\[ R_0 \]

\[ h \]

\[ R_0 \]

\[ |h\rangle \]

\[ |h\rangle \]

\[ R_0 \]

\[ h \]

The precision of the “measurement” depends on the width of the measurement apparatus’ wave function.
Identically prepared molecules

To begin, all molecules are assumed to be in the same state
\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle: \]

\[ |\Psi_N\rangle = (\alpha |0\rangle + \beta |1\rangle)^\otimes N \]
Identically prepared molecules

To begin, all molecules are assumed to be in the same state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$|\Psi_N\rangle = (\alpha|0\rangle + \beta|1\rangle)^\otimes N$$

$$= \sum_{X=0}^{2^N-1} \left( \alpha^N (1-|X|) \beta^N |X| \right) |X\rangle$$
Identically prepared molecules

To begin, all molecules are assumed to be in the same state
\[ |\psi\rangle = \alpha|0\rangle + \beta|1\rangle : \]
\[
|\Psi_N\rangle = (\alpha|0\rangle + \beta|1\rangle)^\otimes N
= \sum_{X=0}^{2^N-1} (\alpha^N(1-|X|) \beta^N|X|) |X\rangle
= \sum_h \left( \alpha^N(1-h) \beta^Nh \right) \sum_{X:|X|=h} |X\rangle
\]
Identically prepared molecules

To begin, all molecules are assumed to be in the same state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$|\Psi_N\rangle = (\alpha|0\rangle + \beta|1\rangle)^\otimes N$$

$$= \sum_{X=0}^{2^N-1} \left( \alpha^N(1-|X|) \beta^N|X| \right) |X\rangle$$

$$= \sum_h \left( \alpha^N(1-h) \beta^Nh \right) \sum_{X:|X|=h} |X\rangle$$

$$= \sum_h \left( \alpha^N(1-h) \beta^Nh \right) \sqrt{\binom{N}{Nh}} |h\rangle$$

The normalized Hamming vectors: $|h\rangle = \left( \frac{N!}{Nh!} \right)^{-\frac{1}{2}} \sum_{X:|X|=h} |X\rangle$. 
Infinite sample

**Frequency operator:** \( F^{(N)} = \sum_{h} hQ^{(N)}_h. \)

The eigenvalues of \( F^{(N)} \) indicate the relative occupation number of \(|1\rangle\).

\[
\lim_{N \to \infty} \| F^{(N)} | \Psi_N \rangle - |\beta \rangle^2 | \Psi_N \rangle \| = 0
\]

If we measure all \( N \) qubits in the \(|0\rangle, |1\rangle\) basis, we find a fraction of \(|1\rangle\) equal to \(|\beta \rangle^2\) with probability one. (Strong law of large numbers.)

Misleading for finite \( N \):

- Measurement outcome with probability 1 \( \Rightarrow \) no disturbance.
- \( |\Psi_N \rangle \) is "close" to an eigenstate of \( F^{(N)} \) for finite \( N \)?
Infinite sample

- Frequency operator: \( F^{(N)} = \sum_h hQ_h^{(N)} \).
- The eigenvalues of \( F^{(N)} \) indicate the relative occupation number of \( |1\rangle \).
- Finkelstein, Hartle, Graham, Farhi, Goldstone, and Gutmann

\[
\lim_{N \to \infty} \left\| F^{(N)} |\Psi_N\rangle - |\beta|^2 |\Psi_N\rangle \right\| = 0
\]
Infinite sample

Frequency operator: \[ F^{(N)} = \sum_h hQ_h^{(N)}. \]

The eigenvalues of \( F^{(N)} \) indicate the relative occupation number of \( |1\rangle \).

Finkelstein, Hartle, Graham, Farhi, Goldstone, and Gutmann

\[
\lim_{N \to \infty} \| F^{(N)} |\Psi_N\rangle - |\beta|^2 |\Psi_N\rangle \| = 0
\]

If we measure all \( N \) qubits in the \( |0\rangle, |1\rangle \) basis, we find a fraction of \( |1\rangle \) equal to \( |\beta|^2 \) with probability one. (Strong law of large numbers.)
Infinite sample

- Frequency operator: \( F^{(N)} = \sum_h hQ_h^{(N)} \).
- The eigenvalues of \( F^{(N)} \) indicate the relative occupation number of \( |1\rangle \).
- Finkelstein, Hartle, Graham, Farhi, Goldstone, and Gutmann

\[
\lim_{N \to \infty} \| F^{(N)} |\Psi_N \rangle - |\beta|^2 |\Psi_N \rangle \| = 0
\]

- If we measure all \( N \) qubits in the \( |0\rangle, |1\rangle \) basis, we find a fraction of \( |1\rangle \) equal to \( |\beta|^2 \) with probability one. (Strong law of large numbers.)
- Misleading for finite \( N \):
  - Measurement outcome with probability 1 \( \Rightarrow \) no disturbance.
  - \( |\Psi_N \rangle \) is "close" to an eigenstate of \( F^{(N)} \) for finite \( N \)?
Post-measurement states

Initial state: \( \rho_N = |\Psi_N\rangle\langle\Psi_N| = (|\psi\rangle\langle\psi|)^\otimes N. \)
Post-measurement states

- Initial state: \( \rho_N = |\Psi_N\rangle\langle\Psi_N| = (|\psi\rangle\langle\psi|)^{\otimes N} \).

- Conditional post-measurement state given outcome \( \ell \):

\[
\rho_N |\ell\rangle = \frac{\tilde{Q}^{(N)}_\ell \rho_N \tilde{Q}^{(N)}_\ell}{P(\tilde{Q}^{(N)}_\ell | \rho_N)}
\]
Post-measurement states

- Initial state: \( \rho_N = |\Psi_N\rangle\langle\Psi_N| = (|\psi\rangle\langle\psi|)^\otimes N \).

- Conditional post-measurement state given outcome \( \ell \):
  \[
  \rho_{N|\ell} = \frac{\tilde{Q}^{(N)}_{\ell} \rho_N \tilde{Q}^{(N)}_{\ell}}{P(\tilde{Q}^{(N)}_{\ell} | \rho_N)}
  \]

- Average post-measurement state:
  \[
  \rho'_N = \int \rho_{N|\ell} P(\tilde{Q}^{(N)}_{\ell} | \rho_N) d\ell = \int \tilde{Q}^{(N)}_{\ell} \rho_N \tilde{Q}^{(N)}_{\ell} d\ell
  \]
Post-measurement states

- Initial state: \( \rho_N = |\Psi_N\rangle\langle\Psi_N| = (|\psi\rangle\langle\psi|)^\otimes N \).

- Conditional post-measurement state given outcome \( \ell \):
  \[
  \rho_N|\ell = \frac{\tilde{Q}_{\ell}^{(N)} \rho_N \tilde{Q}_{\ell}^{(N)}}{P(\tilde{Q}_{\ell}^{(N)} | \rho_N)}
  \]

- Average post-measurement state:
  \[
  \rho'_N = \int \rho_N|\ell P(\tilde{Q}_{\ell}^{(N)} | \rho_N) d\ell = \int \tilde{Q}_{\ell}^{(N)} \rho_N \tilde{Q}_{\ell}^{(N)} d\ell
  \]

- Conditional post measurement state of a single qubit:
  \( \rho_1|\ell = Tr_{N-1} \{\rho_N|\ell\} \).
Post-measurement states

Initial state: \( \rho_N = |\Psi_N\rangle\langle\Psi_N| = (|\psi\rangle\langle\psi|)^\otimes N. \)

Conditional post-measurement state given outcome \( \ell \):

\[
\rho_{N|\ell} = \frac{\tilde{Q}_\ell^{(N)} \rho_N \tilde{Q}_\ell^{(N)}}{P(\tilde{Q}_\ell^{(N)} | \rho_N)}
\]

Average post-measurement state:

\[
\rho'_N = \int \rho_{N|\ell} P(\tilde{Q}_\ell^{(N)} | \rho_N) d\ell = \int \tilde{Q}_\ell^{(N)} \rho_N \tilde{Q}_\ell^{(N)} d\ell
\]

Conditional post measurement state of a single qubit:

\[
\rho_{1|\ell} = Tr_{N-1}\{\rho_{N|\ell}\}.
\]

Average post-measurement state of a single qubit:

\[
\rho'_1 = Tr_{N-1}\{\rho'_N\}.
\]
State disturbance

- Perfect measurements, i.e. $\sigma = 0$.
- Post-measurement $F(\rho_N, \rho'_N) = \langle \Psi_N | \rho'_N | \Psi_N \rangle \leq \frac{1}{\sqrt{2\pi N} |\alpha| |\beta|}$.
State disturbance

- Perfect measurements, i.e. $\sigma = 0$.
- Post-measurement $F(\rho_N, \rho'_{N}) = \langle \Psi_N | \rho'_{N} | \Psi_N \rangle \leq \frac{1}{\sqrt{2\pi N} |\alpha| \cdot |\beta|}$. 
- Conditional post-measurement state: concavity of fidelity.
State disturbance

Perfect measurements, i.e. $\sigma = 0$.

Post-measurement

$$F(\rho_N, \rho'_N) = \langle \Psi_N | \rho'_N | \Psi_N \rangle \leq \frac{1}{\sqrt{2\pi N}} |\alpha| \cdot |\beta|.$$

Conditional post-measurement state: concavity of fidelity.

Conditional single qubit post-measurement:

$$\rho_1|h = (1 - h)|0\rangle\langle 0| + h|1\rangle\langle 1|$$

the qubit is decohered and has diagonal coefficients equal to the measured empirical probability distribution.
State disturbance

Perfect measurements, i.e. $\sigma = 0$.

Post-measurement $F(\rho_N, \rho'_N) = \langle \Psi_N | \rho'_N | \Psi_N \rangle \leq \frac{1}{\sqrt{2\pi N |\alpha| |\beta|}}$.

Conditional post-measurement state: concavity of fidelity.

Conditional single qubit post-measurement:

$$\rho_1|h = (1 - h)|0\rangle\langle 0| + h|1\rangle\langle 1|$$

the qubit is decohered and has diagonal coefficients equal to the measured empirical probability distribution.

With high probability, $1 - h \approx |\alpha|^2$ and $h \approx |\beta|^2$. (Typical sequence theorem.)
State disturbance

- Perfect measurements, i.e. $\sigma = 0$.
- Post-measurement $F(\rho_N, \rho'_N) = \langle \Psi_N | \rho'_N | \Psi_N \rangle \leq \frac{1}{\sqrt{2\pi N |\alpha| \cdot |\beta|}}$.
- Conditional post-measurement state: concavity of fidelity.
- Conditional single qubit post-measurement:
  \[
  \rho_1|h = (1 - h) |0\rangle\langle 0| + h |1\rangle\langle 1|
  \]
the qubit is decohered and has diagonal coefficients equal to the measured empirical probability distribution.
- With high probability, $1 - h \approx |\alpha|^2$ and $h \approx |\beta|^2$. (Typical sequence theorem.)
- Average state of a single qubit $\rho'_1 = |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$. 
Finite accuracy

Assume Gaussian smoothing, for general case, see quant-ph/0403212.

\[
F(\rho_N, \rho'_N) = \langle \Psi_N | \rho'_N | \Psi_N \rangle = \sum_{hh'} b(h) b(h') \int \sqrt{q_h(\ell)q_{\ell}(h')} d\ell
\]

where \( b(h) = \binom{N}{Nh} |\alpha|^{2N(1-h)} |\beta|^{2Nh} \).
Finite accuracy

Assume Gaussian smoothing, for general case, see quant-ph/0403212.

\[ F(\rho_{N}, \rho'_{N}) = \langle \Psi_{N} | \rho'_{N} | \Psi_{N} \rangle \]
\[ = \sum_{hh'} b(h)b(h') \int \sqrt{q_h(\ell)q_{\ell}(h')} d\ell \]

where \( b(h) = \binom{N}{Nh}|\alpha|^{2N(1-h)}|\beta|^{2Nh} \).

Truncate the sum to \( h, h' \in D = [[|\beta|^2 - \Delta, |\beta|^2 + \Delta]] \).
Finite accuracy

Assume Gaussian smoothing, for general case, see quant-ph/0403212.

\[ F(\rho_N, \rho'_N) = \langle \Psi_N | \rho'_N | \Psi_N \rangle = \sum_{hh'} b(h) b(h') \int \sqrt{q_h(\ell)q_{\ell}(h')} d\ell \]

where \( b(h) = \binom{N}{Nh} |\alpha|^{2N(1-h)} |\beta|^{2Nh} \).

- Truncate the sum to \( h, h' \in \mathcal{D} = [|\beta|^2 - \Delta, |\beta|^2 + \Delta] \).
- On this domain, \( \int \sqrt{q_h(\ell)q_{\ell}(h')} d\ell \geq \exp\{-\Delta^2/2\sigma^2\} \).
Finite accuracy

Assume Gaussian smoothing, for general case, see quant-ph/0403212.

\[ F(\rho_N, \rho_N') = \langle \Psi_N | \rho_N' | \Psi_N \rangle = \sum_{h, h'} b(h) b(h') \int \sqrt{q_h(\ell)q_{\ell}(h')} d\ell \]

where \( b(h) = \binom{N}{Nh} |\alpha|^{2N(1-h)} |\beta|^{2Nh} \).

- Truncate the sum to \( h, h' \in D = [|\beta|^2 - \Delta, |\beta|^2 + \Delta] \).
- On this domain, \( \int \sqrt{q_h(\ell)q_{\ell}(h')} d\ell \geq \exp\{-\Delta^2/2\sigma^2\} \).
- \( F(\rho_N, \rho_N') \geq \exp\{-\Delta^2/2\sigma^2\} \left( \sum_{h \in D} b(h) \right)^2 \geq \exp\{-\Delta^2/2\sigma^2\}(1 - e^{-N\Delta^2})^2 \) by the typical sequence theorem.
Finite accuracy

The bound $F(\rho_N, \rho'_N) \geq \exp\{-\Delta^2/2\sigma^2\}(1 - e^{-N\Delta^2})^2$ holds for any $\Delta$. As the size of the sample increases, the measurement accuracy $\sigma$ can decrease as $\frac{1}{\sqrt{N}}$ while maintaining a constant fidelity $F(\rho_N, \rho'_N) \geq 1 - \epsilon$. For a constant measurement accuracy $\sigma$, the fidelity approaches unity as $1 - \ln(N)/N$. Fidelity goes to 1 for $\sigma \propto N^{1-s/2}$ with $0 < s < 1/2$. 
Finite accuracy

- The bound \( F(\rho_N, \rho'_N) \geq \exp\{-\Delta^2 / 2\sigma^2\}(1 - e^{-N\Delta^2})^2 \) holds for any \( \Delta \).

- Optimize over \( \Delta \) to get the tightest bound

\[
F(\rho_N, \rho'_N) \geq 1 - \frac{1 + \ln(2N\sigma^2)}{2N\sigma^2}
\]
Finite accuracy

- The bound $F(\rho_N, \rho_N') \geq \exp\{-\Delta^2/2\sigma^2\}(1 - e^{-N\Delta^2})^2$ holds for any $\Delta$.
- Optimize over $\Delta$ to get the tightest bound

$$F(\rho_N, \rho_N') \geq 1 - \frac{1 + \ln(2N\sigma^2)}{2N\sigma^2}$$

- As the size of the sample increases, the measurement accuracy $\sigma$ can decrease as $1/\sqrt{N}$ while maintaining a constant fidelity $F(\rho_N, \rho_N') \geq 1 - \epsilon$. 

Finite accuracy

- The bound \( F(\rho_N, \rho'_N) \geq \exp\{-\Delta^2/2\sigma^2\}(1 - e^{-N\Delta^2})^2 \) holds for any \( \Delta \).

- Optimize over \( \Delta \) to get the tightest bound

\[
F(\rho_N, \rho'_N) \geq 1 - \frac{1 + \ln(2N\sigma^2)}{2N\sigma^2}
\]

- As the size of the sample increases, the measurement accuracy \( \sigma \) can decrease as \( 1/\sqrt{N} \) while maintaining a constant fidelity \( F(\rho_N, \rho'_N) \geq 1 - \epsilon \).

- For a constant measurement accuracy \( \sigma \), the fidelity approaches unity as \( 1 - \frac{\ln(N)}{N} \).
Finite accuracy

The bound \( F(\rho_N, \rho_N') \geq \exp\{-\Delta^2/2\sigma^2\}(1 - e^{-N\Delta^2})^2 \) holds for any \( \Delta \).

Optimize over \( \Delta \) to get the tightest bound

\[
F(\rho_N, \rho_N') \geq 1 - \frac{1 + \ln(2N\sigma^2)}{2N\sigma^2}
\]

As the size of the sample increases, the measurement accuracy \( \sigma \) can decrease as \( 1/\sqrt{N} \) while maintaining a constant fidelity \( F(\rho_N, \rho_N') \geq 1 - \epsilon \).

For a constant measurement accuracy \( \sigma \), the fidelity approaches unity as \( 1 - \frac{\ln(N)}{N} \).

Fidelity goes to 1 for \( \sigma \propto N^{-s} \) with \( 0 < s < 1/2 \).
Finite accuracy

Similar bound hold for any smooth (Lipschitz continuity) smoothing function \( q_h(\ell) \).
Finite accuracy

- Similar bound hold for any smooth (Lipschitz continuity) smoothing function \( q_h(\ell) \).

- For smooth \( q_h(\ell) \), the fidelity to the conditional post-measurement state \( \rho_{N|\ell} \) goes to one, except with a probability that goes to zero.
Finite accuracy

- Similar bound hold for any smooth (Lipschitz continuity) smoothing function $q_h(\ell)$.
- For smooth $q_h(\ell)$, the fidelity to the conditional post-measurement state $\rho_N|_\ell$ goes to one, except with a probability that goes to zero.
- The construction also holds for initial mixed product state $\rho_N = \nu \otimes \nu \otimes \ldots \otimes \nu$. 

QIP CIAR Meeting, Vancouver, May 2004 – p.17
Finite accuracy

- Similar bound hold for any smooth (Lipschitz continuity) smoothing function $q_h(\ell)$.

- For smooth $q_h(\ell)$, the fidelity to the conditional post-measurement state $\rho_N|\ell$ goes to one, except with a probability that goes to zero.

- The construction also holds for initial mixed product state $\rho_N = \nu \otimes \nu \otimes \ldots \otimes \nu$.

- The scaling $\sigma \propto 1/\sqrt{N}$ is optimum. For $\sigma \ll 1/\sqrt{N}$:
  
  $$F(\rho_N, \rho'_N) = \sum_{hh'} b(h) b(h') \int \sqrt{q_h(\ell)q_{\ell}(h')} d\ell \leq 2\sigma/\sqrt{N\pi}.$$
Let’s not jump to conclusions too fast...
Let’s not jump to conclusions too fast...

If the field is initially in a **mixed** state, e.g.

\[
\rho_F \propto \int e^{-\frac{q^2}{2\sigma^2}} |\phi_q\rangle\langle\phi_q| dq \quad \text{where} \quad |\phi_q\rangle \propto \int e^{-\frac{(r-q)^2}{4\lambda^2}} |r\rangle dr,
\]

we get a **non-ideal** measurement (more than one Krauss operator).
Let's not jump to conclusions too fast...

- If the field is initially in a **mixed** state, e.g.

\[
\rho_F \propto \int e^{-\frac{q^2}{2\sigma^2}} |\phi_q\rangle\langle\phi_q| dq \quad \text{where} \quad |\phi_q\rangle \propto \int e^{-\frac{(r-q)^2}{4\lambda^2}} |r\rangle dr,
\]

we get a **non-ideal** measurement (more than one Krauss operator).

- Measurement accuracy is governed by \(\sigma\).
Let’s not jump to conclusions too fast...

- If the field is initially in a **mixed** state, e.g.

\[ \rho_F \propto \int e^{-\frac{q^2}{2\sigma^2}} |\phi_q\rangle\langle\phi_q| dq \]

where

\[ |\phi_q\rangle \propto \int e^{-\frac{(r-q)^2}{4\lambda^2}} |r\rangle dr, \]

we get a **non-ideal** measurement (more than one Krauss operator).

- Measurement accuracy is governed by \( \sigma \).
- State disturbance is governed by \( \lambda \).
Let’s not jump to conclusions too fast...

- If the field is initially in a **mixed** state, e.g.

\[ \rho_F \propto \int e^{-\frac{q^2}{2\sigma^2}} |\phi_q\rangle\langle \phi_q| dq \ 	ext{where} \ |\phi_q\rangle \propto \int e^{-\frac{(r-q)^2}{4\lambda^2}} |r\rangle dr, \]

we get a **non-ideal** measurement (more than one Krauss operator).

- Measurement accuracy is governed by \( \sigma \).
- State disturbance is governed by \( \lambda \).

- Our tradeoff was established in the “best case scenario”. Heating up the measurement coil will increase the state disturbance and the measurement coarseness.
Mixing up the initial states of the measurement apparatus is equivalent to mixing up the measurement outputs.

State disturbance is governed by the quantum width of individual states of the mixture.

Measurement accuracy is governed by the classical width of the mixture.
Is NMR QIP quantum mechanical?

- Making measurements **throughout the computation** does not alter the final statistics.
Is NMR QIP quantum mechanical?

- Making measurements **throughout the computation** does not alter the final statistics.
- Alternative argument (common in condensed matter physics...) 
  
  $a, b$ and $c = [a, b]$. $|a|, |b|, |c| \approx 1$. 

<table>
<thead>
<tr>
<th>$A_N$</th>
<th>$B_N$</th>
<th>$C_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_N$, $B_N$ =</td>
<td>$1$</td>
<td></td>
</tr>
</tbody>
</table>

Commuting observables behave classically: the act of measuring one does not influence the statistics of others. There is no conflict with what was established here: Disturbing the state $\rightarrow$ altering statistics of all measurements.
Is NMR QIP quantum mechanical?

- Making measurements **throughout the computation** does not alter the final statistics.

- Alternative argument (common in condensed matter physics...)
  
  \[ a, b \text{ and } c = [a, b]. \ |a|, |b|, |c| \approx 1. \]

\[ A_N = \frac{1}{N} \sum_k a(k), \text{ and similarly for } B_N \text{ and } C_N. \]

\[ |A_N|, |B_N|, |C_N| \approx 1. \]
Is NMR QIP quantum mechanical?

- Making measurements throughout the computation does not alter the final statistics.
- Alternative argument (common in condensed matter physics...)  
  \( a, b \) and \( c = [a, b] \).  
  \(|a|, |b|, |c| \approx 1\).

  \[ A_N = \frac{1}{N} \sum_k a_k, \text{ and similarly for } B_N \text{ and } C_N. \]
  \(|A_N|, |B_N|, |C_N| \approx 1\).

  \[ [A_N, B_N] = \frac{1}{N} C_N: \text{ Average observables commute in the limit of large } N! \text{ I.e. } |[A_N, B_N]| \approx \frac{1}{N}. \]
Is NMR QIP quantum mechanical?

- Making measurements throughout the computation does not alter the final statistics.
- Alternative argument (common in condensed matter physics...)

  \[ a, b \text{ and } c = [a, b]. \quad |a|, |b|, |c| \approx 1. \]

  \[ A_N = \frac{1}{N} \sum_k a(k), \text{ and similarly for } B_N \text{ and } C_N. \]

  \[ |A_N|, |B_N|, |C_N| \approx 1. \]

  \[ [A_N, B_N] = \frac{1}{N} C_N: \text{ Average observables commute in the limit of large } N! \text{ I.e. } |[A_N, B_N]| \approx \frac{1}{N}. \]

- Commuting observables behave classically: the act of measuring one does not influence the statistics of others.
Is NMR QIP quantum mechanical?

Making measurements throughout the computation does not alter the final statistics.

Alternative argument (common in condensed matter physics...)

\(a, b\) and \(c = [a, b]\). \(|a|, |b|, |c| \approx 1\).

\[
A_N = \frac{1}{N} \sum_k a^{(k)}, \text{ and similarly for } B_N \text{ and } C_N.
\]

\(|A_N|, |B_N|, |C_N| \approx 1\).

\[
[A_N, B_N] = \frac{1}{N} C_N: \text{ Average observables commute in the limit of large } N! \text{ i.e. } |[A_N, B_N]| \approx \frac{1}{N}.
\]

Commuting observables behave classically: the act of measuring one does not influence the statistics of others.

There is no conflict with what was established here:
Making measurements throughout the computation does not alter the final statistics.

Alternative argument (common in condensed matter physics...) $a$, $b$ and $c = [a, b]$. $|a|, |b|, |c| \approx 1$.

$A_N = \frac{1}{N} \sum_k a(k)$, and similarly for $B_N$ and $C_N$. $|A_N|, |B_N|, |C_N| \approx 1$.

$[A_N, B_N] = \frac{1}{N} C_N$: Average observables commute in the limit of large $N$! I.e. $|[A_N, B_N]| \approx \frac{1}{N}$.

Commuting observables behave classically: the act of measuring one does not influence the statistics of others.

There is no conflict with what was established here:

Disturbing the state $\not\Rightarrow$ altering statistics of all measurements.
The argument of the previous slide suggests that the exact measurement of macroscopic observables might not affect the statistics of other macroscopic observables.
Is NMR QIP quantum mechanical?

- The argument of the previous slide suggests that the exact measurement of macroscopic observables might **not** affect the statistics of other macroscopic observables.

- However, we have seen that these measurements induce **complete decoherence** to the **reduced** state of a single molecule: \( \nu \rightarrow \sum_j \langle x_j | \nu | x_j \rangle | x_j \rangle \langle x_j |. \)
The argument of the previous slide suggests that the exact measurement of macroscopic observables might not affect the statistics of other macroscopic observables.

However, we have seen that these measurements induce complete decoherence to the reduced state of a single molecule:

\[ \nu \rightarrow \sum_j \langle x_j | \nu | x_j \rangle | x_j \rangle \langle x_j |. \]

Hence, if the conventional argument is correct, the apparent quantumness of NMR could be a classical parallelism among the molecules of the sample.
Conclusion

- The exact measurement of a macroscopic observable greatly disturbs the state of an ensemble of identically prepared systems.

- If the measurement is of coarseness $\sigma \gg 1/\sqrt{N}$, the measurement leaves the state essentially unaffected.

- NMR-like measurements may not follow the optimal coarseness-disturbance tradeoff as the state of the coil is generally mixed.

- While coarseness $\sigma \gg 1/\sqrt{N}$ is necessary to leave the state unchanged, it might not be necessary to leave macroscopic observables’ statistics unchanged.

- If this is right, NMR QIP speed-up can be explained by classical parallelism.